ON THE RELATIONSHIP BETWEEN THE WORST-CASE OPTIMIZATION-BASED AND PROBABILITY-CONSTRAINED APPROACHES TO ROBUST ADAPTIVE BEAMFORMING

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ABSTRACT

In this paper, an interesting relationship between the the worst-case optimization-based and probability-constrained approaches to the robust adaptive beamformer design is found both in the cases of Gaussian and non-Gaussian steering vector mismatch. The established relationship demonstrates that the probabilistic beamformer design may be approximately interpreted in terms of the worst-case design, and quantifies the parameters of the latter design in terms of the beamformer outage probability.

***Index Terms*—** Robust adaptive beamforming, probabilistic constraints, worst-case performance optimization

1. INTRODUCTION

A recent popular approach to the designing robust adaptive beamformers is based on the worst-case performance optimization [1]-[5]. The techniques developed using this approach aim to optimize the output signal-to-interference-plus-noise ratio (SINR) for the worst operational conditions. However, the actual worst operational conditions may occur in practice with a rather low probability. To provide more flexibility in the beamformer design, a probability-constrained approach to robust adaptive beamforming has been developed in [6]-[7]. The key idea of the latter approach is to maintain the beamformer distortionless response only for operational conditions which occur with a sufficiently high probability rather than for all operational conditions corresponding to the uncertainty set.

In this paper, an interesting relationship between the probability-constrained and the worst-case optimization-based beamformers is derived. Specifically, we show that the probabilistic robust beamformer design of [7] can be approximated by the worst-case design of [1] and [4]. The established relationship leads to a straightforward interpretation of the worst-case design parameters in terms of the beamformer outage probability.

2. BACKGROUND

The output of a narrowband adaptive beamformer can be expressed as

$$y(l) = \boldsymbol{w}^H \boldsymbol{x}(l) \tag{1}$$

where $\boldsymbol{x}(l) = [x_1(l), \ldots, x_M(l)]^T$ is the complex-valued array snapshot vector, $\boldsymbol{w} = [w_1, \ldots, w_M]^T$ is the complex-valued beamformer weight vector, M is the number of array sensors, l is the sample index, and $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and Hermitian transpose, respectively. The snapshot vector can be modeled as

$$\boldsymbol{x}(l) = s(l)\boldsymbol{a} + \boldsymbol{i}(l) + \boldsymbol{n}(l) \tag{2}$$

where $s(l)$ is the desired signal waveform, $\boldsymbol{a}$ is the signal steering vector, and $\boldsymbol{i}(l)$ and $\boldsymbol{n}(l)$ are the interference and noise components, respectively. A traditional approach to optimize the beamformer weight vector is to maximize the "sample" output SINR

$$\mathrm{SINR} = \frac{\sigma_{\mathrm{s}}^2 |\boldsymbol{w}^H \boldsymbol{a}|^2}{\boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w}} \tag{3}$$

where σ_{s}^2 is the signal variance, $\hat{\boldsymbol{R}} = \frac{1}{K}\sum_{k=1}^{K} \boldsymbol{x}(k)\boldsymbol{x}^H(k)$ is the $M \times M$ sample covariance matrix, and K is the training sample size. Then, the problem of maximizing the SINR in (3) can be equivalently written as

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \quad \text{subject to} \quad \boldsymbol{w}^H \boldsymbol{a} = 1. \tag{4}$$

The solution to this problem is referred to as the sample matrix inversion (SMI) minimum variance beamformer.

In practice, the actual signal steering vector $\tilde{\boldsymbol{a}}$ is usually a distorted version of the presumed steering vector $\boldsymbol{a}$. An essential shortcoming of the SMI beamformer is that it is not robust against such a steering vector mismatch. In [1] and [4], the actual steering vector $\tilde{\boldsymbol{a}}$ has been explicitly modeled as

$$\tilde{\boldsymbol{a}} = \boldsymbol{a} + \boldsymbol{\delta} \neq \boldsymbol{a} \tag{5}$$

where $\boldsymbol{\delta}$ denotes an unknown complex-valued vector describing the effect of steering vector errors. It has been assumed in [1] and [4] that $\boldsymbol{\delta}$ is an unknown deterministic vector that is bounded in its Euclidean norm by some known positive constant $\|\boldsymbol{\delta}\| \leq \varepsilon$. Then, the actual signal steering vector $\tilde{\boldsymbol{a}}$ belongs to the following uncertainty region:

$$\mathcal{A}(\varepsilon) \triangleq \{\boldsymbol{c} \,:\, \boldsymbol{c} = \boldsymbol{a} + \boldsymbol{\delta}, \ \ \|\boldsymbol{\delta}\| \leq \varepsilon\} \tag{6}$$

and the design of the robust adaptive beamformer boils down to solving the problem (4) for the worst-case steering vector [1]:

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \quad \text{subject to} \quad \min_{\|\boldsymbol{\delta}\| \leq \varepsilon} |\boldsymbol{w}^H(\boldsymbol{a} + \boldsymbol{\delta})| \geq 1. \tag{7}$$

The worst-case vector $\boldsymbol{a} + \boldsymbol{\delta}$ that satisfies the constraint in (7) can be shown to lie on the boundary of the uncertainty region $\mathcal{A}(\varepsilon)$. The solution to (7) can be interpreted as a diagonally loaded (DL) SMI beamformer with an adaptive DL factor whose value is optimally matched to the uncertainty region [1], [4].

The worst-case design of (7) may be overly conservative, because the actual worst operational conditions may occur in practice with a very low probability. A more flexible approach to robust beamformer design whose idea is to maintain the beamformer distortionless response only for operational conditions which occur with a sufficiently high probability has been proposed in [6]-[7]. The latter approach corresponds to the following optimization problem:

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \quad \text{subject to} \quad \Pr\{|\boldsymbol{w}^H(\boldsymbol{a}+\boldsymbol{\delta})| \geq 1\} \geq p \tag{8}$$

where $\boldsymbol{\delta}$ is assumed to be random, $\Pr\{\cdot\}$ stands for the probability operator, and p is a preselected probability value that is related to the beamformer outage probability as $p = 1 - p_{\text{out}}$.

In general, the knowledge of the probability density function (pdf) of the steering vector mismatch $\boldsymbol{\delta}$ is required to specify the probability operator $\Pr\{\cdot\}$. Two different assumptions on the statistics of the mismatch vector are of particular interest:

- $\boldsymbol{\delta}$ is drawn from a complex circularly symmetric Gaussian distribution with zero mean and covariance matrix $\boldsymbol{C}_{\boldsymbol{\delta}}$ [8], that is,
$$\boldsymbol{\delta} \sim \mathcal{N}_{\mathcal{C}}(\mathbf{0}_M, \boldsymbol{C}_{\boldsymbol{\delta}}) \tag{9}$$
where $\mathbf{0}_M$ denotes the $M \times 1$ vector of zeros;
- $\boldsymbol{\delta}$ is drawn from a complex circularly symmetric unknown distribution with zero mean and known covariance matrix $\boldsymbol{C}_{\boldsymbol{\delta}}$.

The covariance matrix $\boldsymbol{C}_{\boldsymbol{\delta}}$ captures the second-order statistics of the uncertainties in the steering vector. Even though $\boldsymbol{C}_{\boldsymbol{\delta}}$ may be actually non-diagonal, it typically can be approximated by the scaled identity matrix for the simplicity reason [8].

3. GAUSSIAN MISMATCH

We first establish an approximate relationship between the worst-case optimization-based and probability-constrained approaches for the case of Gaussian steering vector mismatch. Let us assume that the steering vector errors are not too large, so that $|\boldsymbol{w}^H\boldsymbol{\delta}| < |\boldsymbol{w}^H\boldsymbol{a}|$ is valid. Then, we have [1]

$$|\boldsymbol{w}^H(\boldsymbol{a}+\boldsymbol{\delta})| \geq |\boldsymbol{w}^H\boldsymbol{a}| - |\boldsymbol{w}^H\boldsymbol{\delta}|\,. \tag{10}$$

From (10), it follows that

$$\Pr\{|\boldsymbol{w}^H(\boldsymbol{a}+\boldsymbol{\delta})| \geq 1\} \geq \Pr\{|\boldsymbol{w}^H\boldsymbol{a}| - |\boldsymbol{w}^H\boldsymbol{\delta}| \geq 1\}\,. \tag{11}$$

The inequality (11) can be used to approximate the constraint in (8). Indeed, according to (11), the latter constraint is always satisfied if

$$\Pr\{|\boldsymbol{w}^H\boldsymbol{a}| - |\boldsymbol{w}^H\boldsymbol{\delta}| \geq 1\} \geq p\,. \tag{12}$$

It can be assumed without any loss of generality that

$$\text{Re}\{\boldsymbol{w}^H\boldsymbol{a}\} \geq 0\,, \quad \text{Im}\{\boldsymbol{w}^H\boldsymbol{a}\} = 0 \tag{13}$$

because the cost function in (8) is unchanged when $\boldsymbol{w}$ undergoes an arbitrary phase rotation [1]. Using the latter assumption and introducing $b \triangleq \boldsymbol{w}^H\boldsymbol{a} - 1$, we can rewrite the left hand side of (12) as $\Pr\left\{|\boldsymbol{w}^H\boldsymbol{\delta}| \leq b\right\}$. Then, it follows that

$$\begin{aligned}&\Pr\{|\boldsymbol{w}^H\boldsymbol{\delta}| \leq b\}\\ &\quad\geq \Pr\{|\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2} \cap |\text{Im}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2}\}\end{aligned} \tag{14}$$

where $\cap$ denotes the set intersection operation.

From (14), we can approximate (strengthen) the constraint (12) by replacing it with the following constraint:

$$\Pr\{|\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2} \cap |\text{Im}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2}\} \geq p\,. \tag{15}$$

As the random variable $\boldsymbol{w}^H\boldsymbol{\delta}$ is circular zero-mean complex Gaussian distributed, its real and imaginary parts $\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\}$ and $\text{Im}\{\boldsymbol{w}^H\boldsymbol{\delta}\}$ are real Gaussian i.i.d., that is,

$$\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\} \sim \mathcal{N}_{\mathcal{R}}\left(\mathbf{0}_M,\ \|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\|^2/2\right) \tag{16}$$

$$\text{Im}\{\boldsymbol{w}^H\boldsymbol{\delta}\} \sim \mathcal{N}_{\mathcal{R}}\left(\mathbf{0}_M,\ \|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\|^2/2\right). \tag{17}$$

Using the latter property together with the fact that any functions of independent random variables are statistically independent [9], we obtain that

$$\begin{aligned}&\Pr\{|\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2} \cap |\text{Im}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2}\}\\ &\quad= \Pr\{|\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2}\}\ \Pr\{|\text{Im}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2}\}\\ &\quad= \left(\Pr\{|\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2}\}\right)^2\\ &\quad= \left(\Pr\{|\text{Im}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2}\}\right)^2\end{aligned} \tag{18}$$

where the last two equalities follow from the fact that $|\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\}|$ and $|\text{Im}\{\boldsymbol{w}^H\boldsymbol{\delta}\}|$ are identically distributed. That is, the constraint in (15) can be equivalently written as

$$\Pr\left\{|\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2}\right\} \geq \sqrt{p}\,. \tag{19}$$

Now, replacing the original constraint in (8) by its strengthened version (19), the optimization problem in (8) can be approximated by

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \quad \text{subject to} \quad \Pr\left\{|\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2}\right\} \geq \sqrt{p}\,. \tag{20}$$

The problem (20) can be further converted into a deterministic equivalent form. Note that for any Gaussian random variable x and any constant c, the probability $\Pr\{x \leq c\}$ can be written as [9]

$$\Pr\{x \leq c\} = \frac{1}{2} + \frac{1}{2}\,\text{erf}\left(\frac{c - \text{E}\{x\}}{\sqrt{2\text{E}\{(x-\text{E}\{x\})^2\}}}\right) \tag{21}$$

where

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt \tag{22}$$

is the normalized error function for the Gaussian distribution. Using (16) and (21), we have

$$\begin{aligned}&\Pr\{|\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\}| \leq b/\sqrt{2}\}\\ &\quad= \Pr\{\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\} \leq b/\sqrt{2}\} - \Pr\{\text{Re}\{\boldsymbol{w}^H\boldsymbol{\delta}\} \leq -b/\sqrt{2}\}\\ &\quad= \frac{1}{2}\left[\text{erf}\left(\frac{b}{\sqrt{2}\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\|}\right) - \text{erf}\left(-\frac{b}{\sqrt{2}\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\|}\right)\right]\\ &\quad= \text{erf}\left(\frac{b}{\sqrt{2}\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\|}\right)\end{aligned} \tag{23}$$

where the last equality holds because the function $\text{erf}(\cdot)$ is odd. Using (23), the constraint in (20) can be written in the following equivalent deterministic form:

$$\text{erf}\left(\frac{\boldsymbol{w}^H\boldsymbol{a} - 1}{\sqrt{2}\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\|}\right) \geq \sqrt{p}$$

or, equivalently,

$$\sqrt{2}\,\mathrm{erf}^{-1}(\sqrt{p})\,\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\| \leq \boldsymbol{w}^H\boldsymbol{a} - 1\,. \quad (24)$$

Hence, the problem (20) may be expressed as

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H\hat{\boldsymbol{R}}\boldsymbol{w} \quad \text{subject to} \quad \sqrt{2}\,\mathrm{erf}^{-1}(\sqrt{p})\,\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\| \leq \boldsymbol{w}^H\boldsymbol{a} - 1. \quad (25)$$

The latter problem can be identified to be a second-order cone programming (SOCP) problem which is exactly equivalent to the worst-case based robust adaptive beamforming problem of [1] provided that $\boldsymbol{C}_{\boldsymbol{\delta}} = \sigma_{\boldsymbol{\delta}}^2\boldsymbol{I}_M$ and

$$\varepsilon = \sigma_{\boldsymbol{\delta}}\sqrt{2}\,\mathrm{erf}^{-1}(\sqrt{p}) = \sigma_{\boldsymbol{\delta}}\sqrt{2}\,\mathrm{erf}^{-1}(\sqrt{1-p_{\mathrm{out}}})\,. \quad (26)$$

Summarizing the results of this section, we have shown that in the case of Gaussian mismatch, the worst-case optimization-based robust adaptive beamformer can be viewed as a strengthened version of the probability-constrained robust beamformer. This conclusion lends support to our expectation of an improved flexibility of the probabilistic designs with respect to the worst-case designs. Equation (26) explicitly quantifies the relationship between the two approaches providing an interpretation of the worst-case design parameter ε in terms of the beamformer outage probability.

4. MISMATCH WITH UNKNOWN DISTRIBUTION

Let us use the approximation (12) of the original constraint in (8), and additionally exploit the fact that the SINR is invariant to any phase rotation of $\boldsymbol{w}$. Therefore, the constraint (12) can be written as

$$\Pr\{|\boldsymbol{w}^H\boldsymbol{\delta}| < b\} \geq p\,, \qquad \mathrm{Im}\{\boldsymbol{w}^H\boldsymbol{a}\} = 0 \quad (27)$$

where the strict inequality in (27) is used for the sake of simplicity of the subsequent derivations.

As the distribution of $\boldsymbol{\delta}$ is unknown, a natural approach would be to consider the worst-case distribution when computing the probability operator in the first constraint of (27). The following theorem (that is proved in [10] and [11] for more general cases) states the main result for the probability operator in the first constraint of (27).

THEOREM 1: For the worst-case distribution of $\boldsymbol{\delta}$, the probability operator $\Pr\{|\boldsymbol{w}^H\boldsymbol{\delta}| < b\}$ can be upper- and lower-bounded by the following semi-definite programming (SDP) problems:

- *Upper bound SDP:*

$$\min_{\boldsymbol{Z},\,\lambda} (1-\lambda) \geq \Pr\{|\boldsymbol{w}^H\boldsymbol{\delta}| < b\}$$
$$\text{subject to} \quad \|\boldsymbol{Z}^{1/2}\boldsymbol{w}\| - b\lambda \geq 0, \quad \boldsymbol{0} \preceq \boldsymbol{Z} \preceq \boldsymbol{C}_{\boldsymbol{\delta}}, \quad 0 \leq \lambda \leq 1 \quad (28)$$

  where $\boldsymbol{Z}$ is an $M \times M$ Hermitian matrix and λ is a scalar.

- *Lower bound SDP:*

$$\max_{\boldsymbol{P},\,\tau} (1 - \mathrm{Tr}\{\boldsymbol{C}_{\boldsymbol{\delta}}\boldsymbol{P}\}) \leq \Pr\{|\boldsymbol{w}^H\boldsymbol{\delta}| < b\}$$
$$\text{subject to} \quad \boldsymbol{P} \succeq \tau\boldsymbol{w}\boldsymbol{w}^H, \;\; b\tau - 1 \geq 0, \;\; \tau \geq 0 \quad (29)$$

  where $\boldsymbol{P}$ is an $M \times M$ Hermitian matrix and τ is a scalar.

We now show that the bounds in Theorem 1 are tight. It can be easily seen that the objective function in (28) is minimized if $\boldsymbol{Z} = \boldsymbol{C}_{\boldsymbol{\delta}}$ and $\lambda = \|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\|/b$ provided that $\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\| < b$. Then, the upper bound on the probability operator $\Pr\{|\boldsymbol{w}^H\boldsymbol{\delta}| < b\}$ is given by the optimal solution of (28) and can be written as

$$\Pr\{|\boldsymbol{w}^H\boldsymbol{\delta}| < b\} \leq 1 - \frac{\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\|}{b}. \quad (30)$$

Furthermore, taking into account that $b > 0$, the optimal solution of (29) is given by $\tau = 1/b$ and $\boldsymbol{P} = \boldsymbol{w}\boldsymbol{w}^H/b$. Inserting this optimal value of $\boldsymbol{P}$ into the objective function of (29), we can express the lower bound for $\Pr\{|\boldsymbol{w}^H\boldsymbol{\delta}| < b\}$ as

$$\Pr\{|\boldsymbol{w}^H\boldsymbol{\delta}| < b\} \geq 1 - \frac{\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\|}{b}\,. \quad (31)$$

Combining (30) and (31), we conclude that if

$$\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\| < b \quad (32)$$

then the upper and lower bounds coincide and, therefore,

$$\Pr\{|\boldsymbol{w}^H\boldsymbol{\delta}| < \boldsymbol{w}^H\boldsymbol{a} - 1\} = 1 - \frac{\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\|}{\boldsymbol{w}^H\boldsymbol{a} - 1}\,. \quad (33)$$

Inserting (33) into the first constraint of (27), we obtain the following equivalent deterministic constraint

$$\frac{1}{1-p}\,\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\| \leq \boldsymbol{w}^H\boldsymbol{a} - 1\,. \quad (34)$$

Note that (34) automatically guarantees that (32) is satisfied. Thus, the condition (32) can be ignored.

Using (34), the probability-constrained problem of (8) can be approximated as

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H\hat{\boldsymbol{R}}\boldsymbol{w} \quad \text{subject to} \quad \frac{1}{1-p}\,\|\boldsymbol{C}_{\boldsymbol{\delta}}^{1/2}\boldsymbol{w}\| \leq \boldsymbol{w}^H\boldsymbol{a} - 1 \quad (35)$$

where (35) represents a strengthened version of (8). The problem (35) belongs to the class of SOCP problems and is equivalent to the worst-case based robust adaptive beamforming problem of [1] if $\boldsymbol{C}_{\boldsymbol{\delta}} = \sigma_{\boldsymbol{\delta}}^2\boldsymbol{I}_M$ and

$$\varepsilon = \sigma_{\boldsymbol{\delta}}/(1-p) = \sigma_{\boldsymbol{\delta}}/p_{\mathrm{out}}\,. \quad (36)$$

Comparing (26) and (36), we observe that

$$\sigma_{\boldsymbol{\delta}}/(1-p) > \sigma_{\boldsymbol{\delta}}\sqrt{2}\mathrm{erf}^{-1}(\sqrt{p}) \quad (37)$$

and, therefore, in the case of unknown distribution of $\boldsymbol{\delta}$, the radius ε of the spherical uncertainty region suggested by the equivalent SOCP problem is larger than in the case of Gaussian steering vector distribution. This can be explained by the fact that in the case of unknown distribution, the probabilistic robust beamformer should protect the performance against the worst-case distribution.

5. SIMULATIONS

In our simulations, we assume a uniform linear array of $M = 10$ omnidirectional sensors spaced half a wavelength apart, and two interfering sources with plane wavefronts and the directions-of-arrival (DOAs) equal to 30° and 50°, respectively. A total of 100 independent Monte-Carlo runs are used to obtain each point.

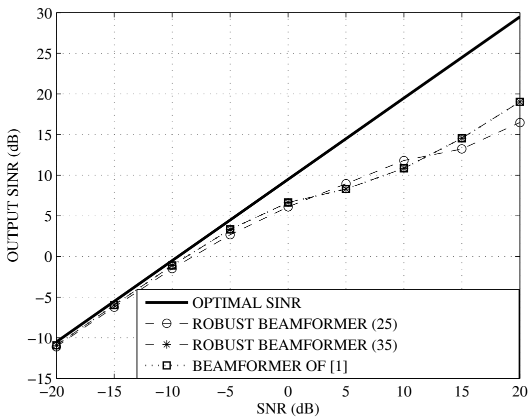

Fig. 1. Output SINR versus SNR. $\mathrm{INR} = 20$ dB, $K = 100$.

A scenario with the Ricean propagation medium is considered where the presumed signal steering vector is a plane wave with the nominal DOA θ_0 while the actual steering vector corresponds to a spatially spread source with the central angle θ_0. The actual mismatch vector $\boldsymbol{\delta}$ can be modeled as [1]

$$\boldsymbol{\delta} = \frac{\sigma_{\boldsymbol{\delta}}}{\sqrt{L}} \sum_{l=1}^{L} e^{j\psi_l} \boldsymbol{a}(\theta_0 + \theta_l) \quad (38)$$

where $\sigma_{\boldsymbol{\delta}}^2$ characterizes the power of scattered nonline-of-sight (NLOS) signal components, L is their number, ψ_l is the phase shift parameter of the lth NLOS component, and θ_l is the angular shift of lth NLOS component with respect to the nominal DOA. The parameters θ_l are independently drawn in each simulation run from a uniform random generator with zero mean and standard deviation of $\sigma_\theta = 5^\circ$. The parameters ψ_l are independently and uniformly drawn from $[0, 2\pi)$ in each run. Throughout this example, $L = 20$, $\theta_0 = 3^\circ$, and $\sigma_{\boldsymbol{\delta}} = 0.3$ have been taken.

Figure 1 compares the beamformers (25), (35) and the worst-case beamformer of [1] with $\varepsilon = 3$ (the latter value is recommended in [1] as a good *ad hoc* choice of this parameter). In this figure, the output SINRs of these three beamformers and the optimal SINR are displayed versus the signal-to-noise ratio (SNR) in the case when $K = 100$, the interference-to-noise ratio (INR) is equal to 20 dB, $p = 0.9$ is taken in (25) and (35), and the non-diagonal matrix $\boldsymbol{C}_{\boldsymbol{\delta}}$ is approximated by $\sigma_{\boldsymbol{\delta}}^2 \boldsymbol{I}_M$ in (25) and (35).

From this figure, we see that all the beamformers tested have quite a similar performance and, in particular, the SINR curves for the beamformer (35) and the technique of [1] are indistinguishable. At high SNRs, the beamformer (25) experiences a slight degradation relative to the other two techniques tested. This additional degradation can be explained by the fact that the random actual mismatch corresponding to the considered Ricean scenario is not Gaussian, while it is assumed to be Gaussian in (25). Note that all the beamformers tested correspond to the same general form of the SOCP problem and, hence, the difference between them is entirely caused by different choices of ε.

6. CONCLUSIONS

We have demonstrated that the popular worst-case optimization-based robust minimum variance beamformer can be viewed as a conservative approximation of the probability-constrained robust minimum variance beamformer. Such a relationship have been found both in the cases of Gaussian and non-Gaussian (unknown) distribution of the steering vector mismatch. The established relationship demonstrates that the probabilistic beamformer designs can be approximately interpreted in the nomenclature of the worst-case designs where the parameters of the latter designs can be quantified in terms of the beamformer outage probability.

7. REFERENCES

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