DETERMINISTIC CRAMÉR-RAO BOUND FOR SYMMETRIC PARAFAC MODEL WITH APPLICATION TO BLIND SPATIAL SIGNATURE ESTIMATION

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ABSTRACT

The symmetric PARAllel FACtor analysis (PARAFAC) model has found numerous applications in array signal processing and communications. In this paper, we derive the deterministic Cramér-Rao Bound (CRB) for the symmetric RARAFAC model and illustrate the obtained results using an example with spatial signature estimation in sensor arrays.

1. INTRODUCTION AND DATA MODEL

A family of blind array processing algorithms including ESPRIT-like method [1]-[2], Second Order Blind Identification (SOBI) algorithm [3]-[4] and blind spatial signature estimation method based on time-varying user power loading [5] exploit the models which essentially share the same structure called a *symmetric* PARAFAC model.

The CRB analysis for the methods based on the symmetric PARAFAC model is of great interest. In this paper, we derive such CRB in a closed form and illustrate the obtained results using an example with spatial signature estimation in sensor arrays.

Let an array of K sensors receive the signals from M narrowband sources. The $K \times 1$ snapshot vector of antenna array outputs can be written as

$$\boldsymbol{y}(n) = \boldsymbol{A}\boldsymbol{s}(n) + \boldsymbol{v}(n) \tag{1}$$

where $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_M]$ is the $K \times M$ complex matrix of the user spatial signatures, $\boldsymbol{a}_m = [a_{1,m}, \ldots, a_{K,m}]^T$ is the $K \times 1$ complex spatial signature of the mth user, $\boldsymbol{s}(n) = [s_1(n), \ldots, s_M(n)]^T$ is the $M \times 1$ complex vector of the user waveforms, $\boldsymbol{v}(n) = [v_1(n), \ldots, v_K(n)]^T$ is the $K \times 1$ vector of additive spatially and temporally white complex Gaussian noise, and $(\cdot)^T$ denotes the transpose. Assuming that there is a block of N snapshots available, the model (1) can be written as

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{S} + \boldsymbol{V} \tag{2}$$

where $\boldsymbol{Y} = [\boldsymbol{y}(1), \ldots, \boldsymbol{y}(N)]$ is the $K \times N$ array data matrix, $\boldsymbol{S} = [\boldsymbol{s}(1), \ldots, \boldsymbol{s}(N)]$ is the $M \times N$ user waveform matrix, and $\boldsymbol{V} = [\boldsymbol{v}(1), \ldots, \boldsymbol{v}(N)]$ is the $K \times N$ sensor noise matrix.

Assuming that the user signals are uncorrelated with each other and sensor noise, the array covariance matrix of the received signals can be written as

$$\boldsymbol{R} = \mathrm{E}\{\boldsymbol{y}(n)\boldsymbol{y}^H(n)\} = \boldsymbol{A}\boldsymbol{Q}\boldsymbol{A}^H + \sigma^2\boldsymbol{I} \tag{3}$$

where $\boldsymbol{Q} = \mathrm{E}\{\boldsymbol{s}(n)\boldsymbol{s}^H(n)\}$ is the diagonal covariance matrix of the signal waveforms, σ^2 is the sensor noise variance, $\boldsymbol{I}$ is the identity matrix, and $(\cdot)^H$ denotes the Hermitian transpose.

2. SYMMETRIC PARAFAC MODEL

Often, it is required to estimate the matrix $\boldsymbol{A}$ in (2) based on the observations $\boldsymbol{Y}$ only. In the multiple user case, this is not possible to do with only one known covariance matrix (3) because the matrix $\boldsymbol{A}$ can be estimated from $\boldsymbol{R}$ only up to an arbitrary unknown unitary matrix. To provide a unique estimate of $\boldsymbol{A}$, several covariance matrices have to be used, see [1]-[5].

In this paper, following the approach of [5] with artificial user power loading, we assume that a set of covariance matrices is obtained by dividing uniformly the whole data block of N snapshots into P sub-blocks, each of $N_s = \lfloor \frac{N}{P} \rfloor$ snapshots, where $\lfloor x \rfloor$ denotes the largest integer less than x. The transmitted power of each user is assumed to be fixed within each particular sub-block while is changed from one sub-block to another. Using such power loading scheme, we obtain that the received snapshots within any pth sub-block correspond to the following covariance matrix

$$\boldsymbol{R}(p) = \boldsymbol{A}\boldsymbol{Q}(p)\boldsymbol{A}^H + \sigma^2\boldsymbol{I} \tag{4}$$

where $\boldsymbol{Q}(p)$ is the diagonal covariance matrix of the user waveforms in the pth sub-block and $p = 1, \ldots, P$.

In practice, the noise power can be estimated and then subtracted from the covariance matrix (4). Let us stack the P matrices $\boldsymbol{R}(p) - \sigma^2\boldsymbol{I}$, $p = 1, \ldots, P$ together to form a three-way array $\underline{\boldsymbol{R}}$. This three-way array has a symmetry dictated by the symmetry of the matrices $\boldsymbol{R}(p) - \sigma^2\boldsymbol{I}$. The (i, l, p)th element of such an array can be written as

$$r_{i,l,p} = [\underline{\boldsymbol{R}}]_{i,l,p} = \sum_{m=1}^{M} a_{i,m}\nu_m(p)a^*_{l,m} \tag{5}$$

where $\nu_m(p) = [\boldsymbol{Q}(p)]_{m,m}$ is the power of the mth user in the pth sub-block and $(\cdot)^*$ denotes the complex conjugate. Defining the $P \times M$ matrix $\boldsymbol{P}$ as

$$\boldsymbol{P} = \begin{bmatrix} \nu_1(1) & \ldots & \nu_M(1) \\ \vdots & \ddots & \vdots \\ \nu_1(P) & \ldots & \nu_M(P) \end{bmatrix} \tag{6}$$

we have that $Q(p) = \mathcal{D}_p\{P\}$ for all $p = 1, \ldots, P$ where $\mathcal{D}_p\{\cdot\}$ is the operator that makes a diagonal matrix by selecting the pth row and putting it on the main diagonal while putting zeros elsewhere.

3. DETERMINISTIC CRAMÉR-RAO BOUND

The model (1) for the nth sample of the pth sub-block can be rewritten as

$$\boldsymbol{y}(p,n) = \boldsymbol{A}\boldsymbol{Q}^{1/2}(p)\tilde{\boldsymbol{s}}(n) + \boldsymbol{v}(n), \qquad n = (p-1)N_s + 1, \ldots, pN_s \tag{7}$$

where $\tilde{\boldsymbol{s}}(n) = [\tilde{s}_1(n), \ldots, \tilde{s}_M(n)]^T = \boldsymbol{Q}^{-1/2}(p)\boldsymbol{s}(n)$ is the vector of normalized signal waveforms and the normalization is done so that all waveforms have unit powers.

Hence, the observations in the pth sub-block satisfy the following model

$$\boldsymbol{y}(p,n) \sim \mathcal{CN}(\boldsymbol{\mu}(p,n), \sigma^2\boldsymbol{I}) \tag{8}$$

where

$$\boldsymbol{\mu}(p,n) = \boldsymbol{A}\boldsymbol{Q}^{1/2}(p)\tilde{\boldsymbol{s}}(n), \quad n = (p-1)N_s + 1, \ldots, pN_s \tag{9}$$

The unknown parameters of the model (7) are all the entries of $\boldsymbol{A}$, the diagonal elements of $\boldsymbol{Q}(p)$ $(p = 1, \ldots, P)$ and the noise power σ^2. Note, however, that the latter parameter is decoupled with the other parameters in the Fisher Information Matrix (FIM) [6]. Therefore, without loss of generality, σ^2 can be excluded from the vector of unknown parameters.

A delicate point regarding the CRB is the inherent permutation and scale ambiguity. To derive a meaningful CRB, we assume that the first row of $\boldsymbol{A}$ is normalized to $[1, \ldots, 1]_{1\times M}$ (this removes the scaling ambiguity), and the first row of $\boldsymbol{P}$ is known and consists of distinct elements (which resolves the permutation ambiguity). Then, the $2(K-1)M \times 1$ real vector of the unknown parameters is given by

$$\boldsymbol{\alpha} = [\boldsymbol{\alpha}_2^T, \ldots, \boldsymbol{\alpha}_K^T]^T \tag{10}$$

where $\boldsymbol{\alpha}_k = [\mathrm{Re}\{\tilde{\boldsymbol{a}}_k\}^T, \mathrm{Im}\{\tilde{\boldsymbol{a}}_k\}^T]^T$ and $\tilde{\boldsymbol{a}}_k = [a_{k,1}, \ldots, a_{k,M}]^T$.

The $(P-1)M \times 1$ vector of nuisance parameters can be expressed as

$$\boldsymbol{\zeta} = [\tilde{\boldsymbol{p}}(2), \ldots, \tilde{\boldsymbol{p}}(P)]^T \tag{11}$$

where $\tilde{\boldsymbol{p}}(p)$ is the pth row of the matrix $\boldsymbol{P}$.

Using (10) and (11), the $(2(K-1)M + (P-1)M) \times 1$ real vector of unknown parameters can be written as

$$\boldsymbol{\theta} = [\boldsymbol{\alpha}^T, \ \boldsymbol{\zeta}^T]^T \tag{12}$$

THEOREM: The $(2(K-1)M + (P-1)M) \times (2(K-1)M + (P-1)M)$ FIM is given by

$$\left[\begin{array}{ccc|ccc} \boldsymbol{J}_{\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2} & & 0 & & & \\ & \ddots & & \boldsymbol{J}_{\boldsymbol{\alpha},\tilde{\boldsymbol{p}}(2)} & \cdots & \boldsymbol{J}_{\boldsymbol{\alpha},\tilde{\boldsymbol{p}}(P)} \\ 0 & & \boldsymbol{J}_{\boldsymbol{\alpha}_K,\boldsymbol{\alpha}_K} & & & \\ \hline & \boldsymbol{J}^T_{\boldsymbol{\alpha},\tilde{\boldsymbol{p}}(2)} & & \boldsymbol{J}_{\tilde{\boldsymbol{p}}(2),\tilde{\boldsymbol{p}}(2)} & & 0 \\ & \vdots & & & \ddots & \\ & \boldsymbol{J}^T_{\boldsymbol{\alpha},\tilde{\boldsymbol{p}}(P)} & & 0 & & \boldsymbol{J}_{\tilde{\boldsymbol{p}}(P),\tilde{\boldsymbol{p}}(P)} \end{array}\right]$$

where

$$\begin{aligned} \boldsymbol{J}_{\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2} &= \cdots = \boldsymbol{J}_{\boldsymbol{\alpha}_K,\boldsymbol{\alpha}_K} \\ &= \frac{2}{\sigma^2}\begin{bmatrix} \mathrm{Re}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} & \mathrm{Im}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} \\ \mathrm{Im}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} & \mathrm{Re}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} \end{bmatrix} \end{aligned} \tag{13}$$

$$\boldsymbol{J}_{\tilde{\boldsymbol{p}}(p),\tilde{\boldsymbol{p}}(p)} = \frac{2}{\sigma^2}\mathrm{Re}\{(\boldsymbol{G}(p))^H\boldsymbol{G}(p)\} \tag{14}$$

$$\boldsymbol{J}_{\boldsymbol{\alpha},\tilde{\boldsymbol{p}}(p)} = \frac{2}{\sigma^2}(\boldsymbol{I}_{K-1} \otimes \tilde{\boldsymbol{F}}(p))\tilde{\boldsymbol{H}}(p) \tag{15}$$

$$\boldsymbol{\Upsilon} = \begin{bmatrix} \boldsymbol{f}_1(1) & \cdots & \boldsymbol{f}_M(1) \\ \vdots & \ddots & \vdots \\ \boldsymbol{f}_1(P) & \cdots & \boldsymbol{f}_M(P) \end{bmatrix} \tag{16}$$

$$\boldsymbol{G}(p) = \begin{bmatrix} \boldsymbol{h}_{1,1}(p) & \cdots & \boldsymbol{h}_{1,M}(p) \\ \vdots & \ddots & \vdots \\ \boldsymbol{h}_{K,1}(p) & \cdots & \boldsymbol{h}_{K,M}(p) \end{bmatrix} \tag{17}$$

$$\tilde{\boldsymbol{F}}(p) = \begin{bmatrix} \mathrm{Re}\{\boldsymbol{F}^H(p)\} & \mathrm{Im}\{\boldsymbol{F}^H(p)\} \\ \mathrm{Im}\{\boldsymbol{F}^H(p)\} & \mathrm{Re}\{\boldsymbol{F}^H(p)\} \end{bmatrix} \tag{18}$$

$$\boldsymbol{F}(p) = [\boldsymbol{f}_1(p), \ldots, \boldsymbol{f}_M(p)] \tag{19}$$

$$\tilde{\boldsymbol{H}}(p) = [\tilde{\boldsymbol{H}}_2^T(p), \ldots, \tilde{\boldsymbol{H}}_K^T(p)]^T \tag{20}$$

$$\tilde{\boldsymbol{H}}_k(p) = \begin{bmatrix} \mathrm{Re}\{\boldsymbol{H}_k(p)\} \\ \mathrm{Im}\{\boldsymbol{H}_k(p)\} \end{bmatrix} \tag{21}$$

$$\boldsymbol{H}_k(p) = [\boldsymbol{h}_{k,1}(p), \ldots, \boldsymbol{h}_{k,M}(p)] \tag{22}$$

$$\begin{aligned} \boldsymbol{f}_m(p) = \Big[&\sqrt{\nu_m(p)}\tilde{s}_m((p-1)N_s + 1), \\ &\ldots, \sqrt{\nu_m(p)}\tilde{s}_m(pN_s)\Big]^T \end{aligned} \tag{23}$$

$$\begin{aligned} \boldsymbol{h}_{k,m}(p) = \Bigg[&\frac{a_{k,m}\tilde{s}_m((p-1)N_s + 1)}{2\sqrt{\nu_m(p)}}, \\ &\ldots, \frac{a_{k,m}\tilde{s}_m(pN_s)}{2\sqrt{\nu_m(p)}}\Bigg]^T \end{aligned} \tag{24}$$

and $\otimes$ denotes the Kronecker matrix product.

The $(K-1)M \times (K-1)M$ spatial signature-related block of the CRB matrix is given in the closed form as

$$\begin{aligned} \mathrm{CRB}_{\boldsymbol{\alpha},\boldsymbol{\alpha}} = \Bigg[&\boldsymbol{J}_{\boldsymbol{\alpha},\boldsymbol{\alpha}} - \frac{2}{\sigma^2}\sum_{p=2}^{P}(\boldsymbol{I}_{K-1} \otimes \tilde{\boldsymbol{F}}(p))\tilde{\boldsymbol{H}}(p) \\ \times &\left[\mathrm{Re}\{\boldsymbol{G}^H(p)\boldsymbol{G}(p)\}\right]^{-1}\tilde{\boldsymbol{H}}^H(p)(\boldsymbol{I}_{K-1} \otimes \tilde{\boldsymbol{F}}(p))^H\Bigg]^{-1} \end{aligned} \tag{25}$$

where the upper-left block of the FIM can be expressed as

$$\boldsymbol{J}_{\boldsymbol{\alpha},\boldsymbol{\alpha}} = \frac{2}{\sigma^2}\boldsymbol{I}_{K-1} \otimes \begin{bmatrix} \mathrm{Re}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} & \mathrm{Im}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} \\ \mathrm{Im}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} & \mathrm{Re}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} \end{bmatrix} \tag{26}$$

PROOF: The (l,k)th element of the FIM is given by [6]

$$\begin{aligned} \mathrm{FIM}_{l,k} &= \frac{2}{\sigma^2} \\ &\times \sum_{p=1}^{P}\sum_{n=(p-1)N_s+1}^{pN_s}\mathrm{Re}\left(\frac{\partial\boldsymbol{\mu}^H(p,n)}{\partial\theta_l}\frac{\partial\boldsymbol{\mu}(p,n)}{\partial\theta_k}\right) \end{aligned} \tag{27}$$

Using (9) along with (27), we have

$$\frac{\partial\mu(p,n)}{\partial\mathrm{Re}\{a_{k,m}\}} = \sqrt{\nu_m(p)}\tilde{s}_m(n)\boldsymbol{e}_k \tag{28}$$

$$\frac{\partial\mu(p,n)}{\partial\mathrm{Im}\{a_{k,m}\}} = j\sqrt{\nu_m(p)}\tilde{s}_m(n)\boldsymbol{e}_k \tag{29}$$

$$\frac{\partial\mu(p,n)}{\partial\nu_m(p)} = \left[\frac{a_{1,m}\tilde{s}_m(n)}{2\sqrt{\nu_m(p)}},\ldots,\frac{a_{K,m}\tilde{s}_m(n)}{2\sqrt{\nu_m(p)}}\right]^T \tag{30}$$

where $\boldsymbol{e}_k$ is the vector containing one in the kth position and zeros elsewhere.

Using (28) and (29) along with (27) we obtain that

$$\begin{aligned} &\boldsymbol{J}_{\mathrm{Re}\{a_{k,m}\},\mathrm{Re}\{a_{k,l}\}} = \boldsymbol{J}_{\mathrm{Im}\{a_{k,m}\},\mathrm{Im}\{a_{k,l}\}} \\ &= \frac{2}{\sigma^2}\sum_{p=1}^{P}\sum_{n=(p\;1)N_s+1}^{pN_s}\mathrm{Re}\left\{\sqrt{\nu_m(p)\nu_l(p)}\tilde{s}_m^*(n)\tilde{s}_l(n)\right\} \\ &= \frac{2}{\sigma^2}\mathrm{Re}\{\boldsymbol{\xi}_m^H\boldsymbol{\xi}_l\} \end{aligned} \tag{31}$$

where $\boldsymbol{\xi}_m = [\boldsymbol{f}_m^T(1),\ldots,\boldsymbol{f}_m^T(P)]^T$.

Similarly,

$$\begin{aligned} \boldsymbol{J}_{\mathrm{Im}\{a_{k,m}\},\mathrm{Re}\{a_{k,l}\}} &= \boldsymbol{J}_{\mathrm{Re}\{a_{k,m}\},\mathrm{Im}\{a_{k,l}\}} \\ &= \frac{2}{\sigma^2}\mathrm{Im}\{\boldsymbol{\xi}_m^H\boldsymbol{\xi}_l\} \end{aligned} \tag{32}$$

Therefore,

$$\begin{aligned} &\boldsymbol{J}_{\mathrm{Re}\{\boldsymbol{\alpha}_k\},\mathrm{Re}\{\boldsymbol{\alpha}_k\}} = \boldsymbol{J}_{\mathrm{Im}\{\boldsymbol{\alpha}_k\},\mathrm{Im}\{\boldsymbol{\alpha}_k\}} \\ &= \frac{2}{\sigma^2}\begin{bmatrix} \mathrm{Re}\{\boldsymbol{\xi}_1^H\boldsymbol{\xi}_1\} & \ldots & \mathrm{Re}\{\boldsymbol{\xi}_1^H\boldsymbol{\xi}_M\} \\ \vdots & \ddots & \vdots \\ \mathrm{Re}\{\boldsymbol{\xi}_M^H\boldsymbol{\xi}_1\} & \ldots & \mathrm{Re}\{\boldsymbol{\xi}_M^H\boldsymbol{\xi}_M\} \end{bmatrix} \\ &= \frac{2}{\sigma^2}\mathrm{Re}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} \end{aligned} \tag{33}$$

and

$$\begin{aligned} &\boldsymbol{J}_{\mathrm{Im}\{\boldsymbol{\alpha}_k\},\mathrm{Re}\{\boldsymbol{\alpha}_k\}} = \boldsymbol{J}_{\mathrm{Re}\{\boldsymbol{\alpha}_k\},\mathrm{Im}\{\boldsymbol{\alpha}_k\}} \\ &= \frac{2}{\sigma^2}\begin{bmatrix} \mathrm{Im}\{\boldsymbol{\xi}_1^H\boldsymbol{\xi}_1\} & \ldots & \mathrm{Im}\{\boldsymbol{\xi}_1^H\boldsymbol{\xi}_M\} \\ \vdots & \ddots & \vdots \\ \mathrm{Im}\{\boldsymbol{\xi}_M^H\boldsymbol{\xi}_1\} & \ldots & \mathrm{Im}\{\boldsymbol{\xi}_M^H\boldsymbol{\xi}_M\} \end{bmatrix} \\ &= \frac{2}{\sigma^2}\mathrm{Im}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} \end{aligned} \tag{34}$$

Using (33) and (34), we obtain (13). Note that the right-hand side of (13) does not depend on the index k. Hence,

$$\begin{aligned} \boldsymbol{J}_{\boldsymbol{\alpha},\boldsymbol{\alpha}} &= \begin{bmatrix} \boldsymbol{J}_{\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{J}_{\boldsymbol{\alpha}_K,\boldsymbol{\alpha}_K} \end{bmatrix} \\ &= \frac{2}{\sigma^2}\boldsymbol{I}_{K\;1}\otimes\begin{bmatrix} \mathrm{Re}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} & \mathrm{Im}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} \\ \mathrm{Im}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} & \mathrm{Re}\{\boldsymbol{\Upsilon}^H\boldsymbol{\Upsilon}\} \end{bmatrix} \end{aligned} \tag{35}$$

Next, using (30) along with (27) we can write for $p = 2,\ldots,P$ and $m,l = 1,\ldots,M$

$$\begin{aligned} &\left[\boldsymbol{J}_{\tilde{\boldsymbol{p}}(p),\tilde{\boldsymbol{p}}(p)}\right]_{m,l} = \frac{2}{\sigma^2} \\ &\times\sum_{n=(p\;1)N_s+1}^{pN_s}\sum_{k=1}^{K}\mathrm{Re}\left\{\frac{(a_{k,m}\tilde{s}_m(n))^*}{2\sqrt{\nu_m(p)}}\frac{a_{k,l}\tilde{s}_l(n)}{2\sqrt{\nu_l(p)}}\right\} \\ &= \frac{2}{\sigma^2}\mathrm{Re}\{\boldsymbol{c}_m^H(p)\boldsymbol{c}_l(p)\} \end{aligned} \tag{36}$$

where $\boldsymbol{c}_m(p) = [\boldsymbol{h}_{1,m}^T(p),\ldots,\boldsymbol{h}_{K,m}^T(p)]^T$. Stacking all M^2 elements given by (36) in one matrix we have for $p = 2,\ldots,P$

$$\begin{aligned} &\boldsymbol{J}_{\tilde{\boldsymbol{p}}(p),\tilde{\boldsymbol{p}}(p)} = \frac{2}{\sigma^2} \\ &\times\begin{bmatrix} \mathrm{Re}\{\boldsymbol{c}_1^H(p)\boldsymbol{c}_1(p)\} & \ldots & \mathrm{Re}\{\boldsymbol{c}_1^H(p)\boldsymbol{c}_M(p)\} \\ \vdots & \ddots & \vdots \\ \mathrm{Re}\{\boldsymbol{c}_M^H(p)\boldsymbol{c}_1(p)\} & \ldots & \mathrm{Re}\{\boldsymbol{c}_M^H(p)\boldsymbol{c}_M(p)\} \end{bmatrix} \\ &= \frac{2}{\sigma^2}\mathrm{Re}\{\boldsymbol{G}^H(p)\boldsymbol{G}(p)\} \end{aligned} \tag{37}$$

Finally, using (28), (29), and (30) along with (27) we can write for $p = 2,\ldots P$; $k = 2,\ldots,K$, and $m,l = 1,\ldots,M$

$$\begin{aligned} &\left[\boldsymbol{J}_{\mathrm{Re}\{\boldsymbol{a}_k\},\tilde{\boldsymbol{p}}(p)}\right]_{m,l} = \frac{2}{\sigma^2} \\ &\times\sum_{n=(p\;1)N_s+1}^{pN_s}\mathrm{Re}\left\{\frac{1}{2}\frac{\sqrt{\nu_m(p)}}{\sqrt{\nu_l(p)}}\tilde{s}_m^*(n)a_{k,l}\tilde{s}_l(n)\right\} \\ &= \frac{2}{\sigma^2}\mathrm{Re}\{\boldsymbol{f}_m^H(p)\boldsymbol{h}_{k,l}(p)\} \end{aligned} \tag{38}$$

$$\begin{aligned} &\left[\boldsymbol{J}_{\mathrm{Im}\{\boldsymbol{a}_k\},\tilde{\boldsymbol{p}}(p)}\right]_{m,l} = \frac{2}{\sigma^2} \\ &\times\sum_{n=(p\;1)N_s+1}^{pN_s}\mathrm{Re}\left\{j\frac{1}{2}\frac{\sqrt{\nu_m(p)}}{\sqrt{\nu_l(p)}}\tilde{s}_m^*(n)a_{k,l}\tilde{s}_l(n)\right\} \\ &= \frac{2}{\sigma^2}\mathrm{Im}\{\boldsymbol{f}_m^H(p)\boldsymbol{h}_{k,l}(p)\} \end{aligned} \tag{39}$$

Collecting all $(K\;1)M^2$ elements given by (38) and $(K\;1)M^2$ elements given by (39) in one matrix, we obtain for $p = 2,\ldots,P$

$$\boldsymbol{J}_{\boldsymbol{\alpha},\tilde{\boldsymbol{p}}(p)} = \frac{2}{\sigma^2}\begin{bmatrix} \begin{bmatrix} \mathrm{Re}\{\boldsymbol{F}^H(p)\boldsymbol{H}_2(p)\} \\ \mathrm{Im}\{\boldsymbol{F}^H(p)\boldsymbol{H}_2(p)\} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \mathrm{Re}\{\boldsymbol{F}^H(p)\boldsymbol{H}_K(p)\} \\ \mathrm{Im}\{\boldsymbol{F}^H(p)\boldsymbol{H}_K(p)\} \end{bmatrix} \end{bmatrix} \tag{40}$$

Observing that

$$\begin{bmatrix} \mathrm{Re}\{\boldsymbol{F}^H(p)\boldsymbol{H}_k(p)\} \\ \mathrm{Im}\{\boldsymbol{F}^H(p)\boldsymbol{H}_k(p)\} \end{bmatrix} = \tilde{\boldsymbol{F}}(p)\tilde{\boldsymbol{H}}_k(p) \tag{41}$$

we can further simplify (40) to

$$\boldsymbol{J}_{\boldsymbol{\alpha},\tilde{\boldsymbol{p}}(p)} = \frac{2}{\sigma^2}\left(\boldsymbol{I}_{K\;1}\otimes\tilde{\boldsymbol{F}}(p)\right)\tilde{\boldsymbol{H}}(p) \tag{42}$$

Also, note that

$$\boldsymbol{J}_{\boldsymbol{\alpha},\tilde{\boldsymbol{p}}(p)}^T = \boldsymbol{J}_{\tilde{\boldsymbol{p}}(p),\boldsymbol{\alpha}} \tag{43}$$

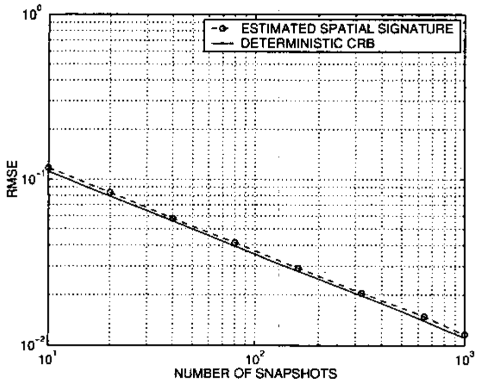

Fig. 1. CRB and RMSE versus N.

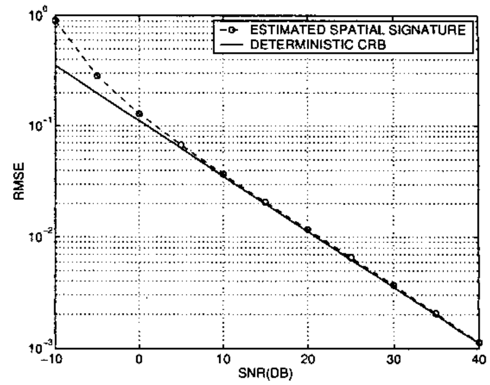

Fig. 2. CRB and RMSE versus SNR.

Using (35), (37), (42) and (43) we obtain the expressions (13)-(24).

Computing the CRB for $\boldsymbol{\theta}$ requires the inverse of the $(2(K\ \ 1)M+(P\ \ 1)M)\times(2(K\ \ 1)M+(P\ \ 1)M)$ FIM matrix. Our objective is to obtain the CRB associated with the vector parameter $\boldsymbol{\alpha}$ only, avoiding the inverse of the full FIM matrix. Exploiting the fact that the lower-right sub-block of the FIM is a block-diagonal matrix and using the partitioned matrix inversion lemma (see [6], p. 572), after some algebra we obtain (25)-(26) and the proof is complete.

4. SIMULATIONS

In order to test the derived CRB we consider a simple example with spatial signature estimation of a single user and assume that the BPSK signal impinges on the linear array of 4 sensors and unknown geometry from $\theta = \ \ 50^\circ$ relative to the broadside direction. It is well known that in the single-user case, a single covariance matrix is sufficient to guarantee the uniqueness of the spatial signature estimate which is given by the principal eigenvector of the sample covariance matrix $\hat{\boldsymbol{R}}$.

We compare the Root-Mean-Square Error (RMSE) performance of such a principal eigenvector-based estimator with the derived CRB. The RMSE is computed as

$$\mathrm{RMSE} = \sqrt{\frac{1}{LK}\sum_{l=1}^{L}\|\hat{\boldsymbol{a}}(l)\ \ \ \boldsymbol{a}\|_F^2}$$

where $L = 100$ is the number of independent simulation runs and $\hat{\boldsymbol{a}}(l)$ is the estimate of $\boldsymbol{a}$ obtained in the lth run. Note that the scaling ambiguity is eliminated by normalizing $\hat{\boldsymbol{a}}(l)$ with respect to the first (reference) sensor. The CRB is computed as

$$\mathrm{CRB} = \sqrt{\frac{1}{K\ \ 1}\mathrm{Tr}\{\mathrm{CRB}_{\alpha,\alpha}\}}$$

Figure 1 displays the RMSE and the CRB versus the number of snapshots N for the Signal-to-Noise Ratio (SNR) equal to 10 dB. Figure 2 shows the same quantities versus the SNR for $N = 100$.

It can be seen that the principal eigenvector-based spatial signature estimator approaches CRB at high SNR. This validates our CRB analysis.

5. CONCLUSIONS

The closed-form expressions for the deterministic CRB for the symmetric PARAFAC model have been derived. The simulation example with blind spatial signature estimation illustrates and validates our CRB analysis.

6. REFERENCES

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