Multiuser Multihop MIMO Relay System Design Based on Mutual Information Maximization

Zhiqiang He, *Member, IEEE*, Sichuan Guo, Yuanbiao Ou, and Yue Rong, *Senior Member, IEEE*

***Abstract*—In this paper, we consider multiuser multihop relay communication systems, where the users, relays, and the destination node may have multiple antennas. We address the issue of source and relay precoding matrices design to maximize the system mutual information (MI). By exploiting the link between the maximal MI and the weighted minimal mean-squared error (WMMSE) objective functions, we show that the intractable maximal MI-based source and relay optimization problem can be solved via the WMMSE-based source and relay design through an iterative approach which is guaranteed to converge to at least a stationary point. For the WMMSE problem, we derive the optimal structure of the relay precoding matrices and show that the WMMSE matrix at the destination node can be decomposed into the sum of WMMSE matrices at all hops. Under a (moderately) high signal-to-noise ratio (SNR) condition, this WMMSE matrix decomposition significantly simplifies the solution to the WMMSE problem. Numerical simulations are performed to demonstrate the effectiveness of the proposed algorithm.**

***Index Terms*—MIMO relay, multiuser, multihop relay, mutual information.**

I. Introduction

MULTIPLE-INPUT MULTIPLE-OUTPUT (MIMO) relay communication technique has attracted much research interest due to its capability in enhancing the system reliability and extending the network coverage [1]–[3]. The relay node can use regenerative or non-regenerative relay strategies [4]. As the distance between source and destination increases, in order to guarantee the system coverage, multiple relay nodes are needed to relay signals from source to destination. In such scenario, non-regenerative MIMO relay systems have been shown to outperform the regenerative ones in computational complexity and system delay [5].

For a single-user multihop MIMO relay system with any number of hops, the optimality of channel diagonalization has been proven in [6]. For a downlink multiuser MIMO relay system where each user is equipped with a single antenna, the source and relay precoding matrices design has been investigated in [7]–[10]. In particular, the upper and lower bounds of the achievable sum rate have been established in [7]. Source and relay matrices that maximize the sum capacity have been studied in [8]. A joint beamforming and power allocation algorithm has been developed in [9] considering the quality-of-service (QoS) constraints. In [10], the mismatch between the true and outdated channel state information (CSI) has been considered in the transceiver design.

In multiuser two-hop relay systems, where the users and the relay node are equipped with multiple antennas, the source and relay precoding matrices maximizing the system mutual information (MI) have been derived in [11] and [12]. In particular, the sum rate maximization and the power minimization problems have been studied in [11], while the weighted sum rate maximization has been considered in [12]. Transceiver designs for two-hop interference MIMO relay systems have been addressed in [13] and [14]. In the multiuser multihop MIMO relay uplink communication system, a simplified algorithm of optimizing the source and relay precoding matrices based on the minimal mean-squared error (MMSE) criterion has been proposed in [15].

In this paper, we focus on multiuser multihop linear non-regenerative (amplify-and-forward) MIMO relay communication systems. Different to the MMSE objective in [15], we aim at maximizing the system MI. The MI maximization problem is more challenging to solve than the MMSE optimization problem. Compared with [7]–[14] which consider only two-hop relay systems, we address multihop multiuser relay systems with any number of hops. Since the MI-based source and relay matrices design problem is intractable to solve, we convert the original problem to weighted MMSE (WMMSE)-based problem by exploiting the link between the MI and WMMSE objectives. We would like to mention that such link was first established for a single-hop MIMO system in [16]. Later on, it has been used in transceiver design for interference MIMO systems [17]. In this paper, we extend the MI-WMMSE link to multihop multiuser MIMO relay systems with any number of hops.

We develop an iterative algorithm to maximize the system MI by solving the WMMSE problem at each iteration. For the WMMSE problem, we derive the structure of the optimal relay precoding matrices and show that the WMMSE matrix at the destination node can be decomposed into the sum of WMMSE matrices at all hops. We would like to mention that the decomposition of the (un-weighted) MMSE matrix was first discovered in [18] for a single-user two-hop MIMO relay system, and was

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Z. He, S. Guo, and Y. Ou are with the Key Laboratory of Universal Wireless Communication, Ministry of Education, Beijing University of Posts and Telecommunications, Beijing 100876, China (e-mail: hezq@bupt.edu.cn; guosichuan@gmail.com; 821211611@qq.com).

Y. Rong is with the Department of Electrical and Computer Engineering, Curtin University, Bentley, WA 6102, Australia (e-mail: y.rong@curtin.edu.au).

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extended to multiuser multihop MIMO relay systems in [15]. Our paper generalizes [15] from un-weighted MMSE matrix decomposition to WMMSE matrix decomposition in multiuser multihop MIMO relay systems with any number of hops and any number of users.

At a (moderately) high signal-to-noise ratio (SNR), the WMMSE matrix decomposition enables the overall WMMSE optimization problem to be decomposed into subproblems of the source precoding matrices optimization and the relay precoding matrices optimization. Such decomposition greatly reduces the complexity of solving the WMMSE problem. In this way, the relay precoding matrices can be optimized successively with the local channel state information (CSI) knowledge and the weight matrix in each iteration. Moreover, we find that the WMMSE problem for optimizing the source precoding matrices is more challenging to solve than the source matrices optimization problem in [15]. Interestingly, we show that this subproblem can be transformed into the WMMSE-based joint transmitter and receiver optimization of a single-hop multiuser MIMO uplink communication system. An iterative algorithm is developed to solve this equivalent problem by updating the transmitter precoding matrices and the receiver matrix alternatingly. In particular, each source precoding matrix can be updated independently using the Lagrange multiplier method.

We would like to note that the algorithms in [20] can be used to solve a general class of non-convex optimization problems in multihop MIMO relay networks including the weighted sum-rate maximization. However, our proposed algorithm can exploit the optimal structure of the relay precoding matrices (in Theorem 2) to reduce the complexity and improve the convergence rate for the special case of sum-rate maximization.

Numerical simulations demonstrate the effectiveness of the proposed algorithms, which typically converge in a few iterations. We would like to note that although we focus on multiaccess MIMO relay systems, the algorithms developed in this paper can be applied to broadcasting MIMO relay systems by exploiting the uplink-downlink duality for multihop linear non-regenerative MIMO relay systems [19]–[22]. For notational convenience, we consider a narrow band single-carrier system in this paper, and our algorithm can be applied in each subcarrier of a broadband multicarrier multihop MIMO relay system.

The rest of the paper is organized as follows. In Section II, we present the model of a linear non-regenerative multiuser multihop MIMO relay communication system. The proposed source and relay precoding matrices design algorithms are developed in Section III. In Section IV, numerical examples are shown to demonstrate the performance of the proposed algorithms. Conclusion are drawn in Section V. The following notations are used throughout the paper: $(\cdot)^T$, $(\cdot)^H$, $(\cdot)^{-1}$, $\mathrm{tr}(\cdot)$, $|\cdot|$ denote the matrix transpose, Hermitian transpose, inversion, trace, and determinant, respectively; $E[\cdot]$ stands for the statical expectation with respect to the signal and noise; $bd(\cdot)$ denotes a block diagonal matrix; $\mathbf{I}_n$ denotes the $n \times n$ identity matrix.

II. System Model and Main Objective

We consider a multiuser multihop MIMO relay communication system as shown in Fig. 1, where N_u users simultaneously transmit information to one destination node via $L-1$ relay nodes in serial. The lth relay node has N_l, $l=1,\ldots,L-1$, antennas, and the destination node has N_L antennas. The ith user is equipped with M_i, $i=1,\ldots,N_u$, antennas. The total number of independent data streams from all users is denoted as $N_0 = \sum_{i=1}^{N_u} M_i$, which should satisfy $N_0 \le \min\{N_1,\ldots,N_L\}$, so that the system can support N_0 active symbols in each transmission. We assume the orthogonality among different hops, as adopted in [6], [20], and [21], meaning that the signal transmitted by the lth relay can only be received by the $(l+1)$-th relay due to the propagation pathloss and proper channel reuse.

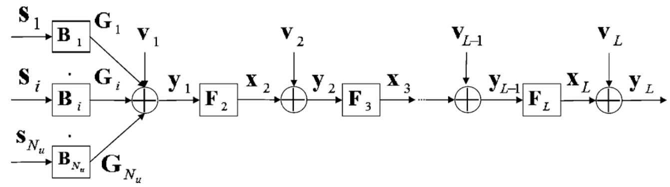

Fig. 1. Block diagram of an N_u-user L-hop linear non-regenerative MIMO relay communication system.

The $M_i \times 1$ source signal vector $\mathbf{s}_i$ is linearly precoded by the $M_i \times M_i$ source precoding matrix $\mathbf{B}_i$. The precoded signal vectors

$$\mathbf{u}_i = \mathbf{B}_i \mathbf{s}_i, \quad i = 1,\ldots,N_u \tag{1}$$

are transmitted to the first relay node. The received signal vector at the first relay node is given by

$$\mathbf{y}_1 = \sum_{i=1}^{N_u} \mathbf{G}_i \mathbf{B}_i \mathbf{s}_i + \mathbf{v}_1 \triangleq \mathbf{H}_1 \mathbf{F}_1 \mathbf{s} + \mathbf{v}_1 \triangleq \mathbf{H}_1 \mathbf{x}_1 + \mathbf{v}_1 \tag{2}$$

where $\mathbf{G}_i$ is the $N_1 \times M_i$ MIMO channel matrix between the ith user and the first relay node, $\mathbf{v}_1$ is the $N_1 \times 1$ independent and identically distributed (i.i.d.) additive white Gaussian noise (AWGN) vector at the first relay node, $\mathbf{s} \triangleq [\mathbf{s}_1^T,\ldots,\mathbf{s}_{N_u}^T]^T$ is the vector of all source signals, $\mathbf{x}_1 = \mathbf{F}_1 \mathbf{s}$ is the signal vector transmitted by all source nodes, and

$$\mathbf{H}_1 \triangleq [\mathbf{G}_1,\ldots,\mathbf{G}_{N_u}], \quad \mathbf{F}_1 \triangleq bd(\mathbf{B}_1,\ldots,\mathbf{B}_{N_u}). \tag{3}$$

Here we assumed that all users are synchronized perfectly. In (3), $\mathbf{H}_1$ stands for the equivalent $N_1 \times N_0$ channel matrix between all users and the first relay node, and $\mathbf{F}_1$ stands for the $N_0 \times N_0$ block diagonal source precoding matrix of all users. We assume that $E[\mathbf{s}\mathbf{s}^H] = \mathbf{I}_{N_0}$.

We adopt the non-regenerative relay strategy as in [6], where each relay node amplifies (linearly precodes) and forwards its received signals. Thus, the relationship between the input and output vectors at the lth relay node is given by

$$\mathbf{x}_{l+1} = \mathbf{F}_{l+1} \mathbf{y}_l, \quad l = 1,\ldots,L-1 \tag{4}$$

where $\mathbf{F}_{l+1}$ is the $N_l \times N_l$ precoding matrix at the lth relay node, and $\mathbf{y}_l$ is the $N_l \times 1$ signal vector received by the lth relay node with

$$\mathbf{y}_l = \mathbf{H}_l \mathbf{x}_l + \mathbf{v}_l, \quad l = 1,\ldots,L-1. \tag{5}$$

Here $\mathbf{H}_l$ is the $N_l \times N_{l-1}$ channel matrix of the lth hop, and $\mathbf{v}_l$ is the $N_l \times 1$ i.i.d. AWGN vector at the lth relay node. The signal vector received at the destination node is given by (5) with $l = L$. We assume that all noises are complex circularly symmetric with zero mean and unit variance.

From (2)–(5), we have

$$\mathbf{y}_l = \mathbf{A}_l \mathbf{s} + \bar{\mathbf{v}}_l, \quad l = 1, \ldots, L \tag{6}$$

where $\mathbf{A}_l$ is the equivalent MIMO channel matrix given by[1]

$$\mathbf{A}_l = \prod_{i=l}^{1} (\mathbf{H}_i \mathbf{F}_i), \quad l = 1, \ldots, L \tag{7}$$

and $\bar{\mathbf{v}}_l$ is the equivalent noise vector whose covariance matrix is

$$\mathbf{C}_1 = \mathbf{I}_{N_1}$$
$$\mathbf{C}_l = \sum_{j=2}^{l} \left(\prod_{i=l}^{j} (\mathbf{H}_i \mathbf{F}_i) \prod_{i=j}^{l} (\mathbf{F}_i^H \mathbf{H}_i^H) \right) + \mathbf{I}_{N_l}, \quad l = 2, \ldots, L.$$

From (1), the transmission power at the ith user is $tr(\mathbf{B}_i \mathbf{B}_i^H)$, $i = 1, \ldots, N_u$. Using (4) and (5), the transmission power consumed by the $(l-1)$-th relay node can be written as

$$tr(E[\mathbf{x}_l \mathbf{x}_l^H]) = tr(\mathbf{F}_l \mathbf{D}_{l-1} \mathbf{F}_l^H), \quad l = 2, \ldots, L \tag{8}$$

where

$$\mathbf{D}_l \triangleq E[\mathbf{y}_l \mathbf{y}_l^H] = \mathbf{A}_l \mathbf{A}_l^H + \mathbf{C}_l, \quad l = 1, \ldots, L-1 \tag{9}$$

is the covariance matrix of $\mathbf{y}_l$.

Our main objective is to find the optimal source precoding matrices $\{\mathbf{B}_i\} \triangleq \{\mathbf{B}_1, \ldots, \mathbf{B}_{N_u}\}$ and relay precoding matrices $\{\mathbf{F}_l\} \triangleq \{\mathbf{F}_2, \ldots, \mathbf{F}_L\}$ to maximize the system MI [23], [6] subjecting to transmission power constraint at the users and the relay nodes, which can be written as

$$\max_{\{\mathbf{B}_i\}, \{\mathbf{F}_l\}} \quad \log \left| \mathbf{I}_{N_0} + \mathbf{A}_L^H \mathbf{C}_L^{-1} \mathbf{A}_L \right| \tag{10}$$
$$\text{s.t.} \quad tr\left(\mathbf{F}_l \mathbf{D}_{l-1} \mathbf{F}_l^H\right) \leq p_l, \quad l = 2, \ldots, L \tag{11}$$
$$tr\left(\mathbf{B}_i \mathbf{B}_i^H\right) \leq q_i, \quad i = 1, \ldots, N_u \tag{12}$$

where p_l is the power available at the $(l-1)$-th relay node and q_i is the power budget at the ith user.

The problem (10)–(12) is highly non-convex with matrix variables. It is computationally intractable to obtain the globally optimal solution, in particular for multihop systems with $L \geq 3$. In the following, we propose simplified algorithms with low computational complexity for the problem (10)–(12) by exploiting the MSE matrix decomposition technique [15], and the link between the maximal MI and the WMMSE objectives [16].

III. Proposed Source and Relay Precoding Matrices Design Algorithms

Let us introduce the MMSE matrix $\mathbf{E}_L$ of the signal waveform estimation at the destination node as [6], [15]

$$\mathbf{E}_L = \left(\mathbf{I}_{N_0} + \mathbf{A}_L^H \mathbf{C}_L^{-1} \mathbf{A}_L\right)^{-1}. \tag{13}$$

[1]Matrix multiplication depends on the order of matrices. Here the lower index is for the first matrix and the upper index is for the last matrix in the multiplication, e.g., for $l > 1$, $\prod_{i=l}^{1} \mathbf{A}_i = \mathbf{A}_l \mathbf{A}_{l-1} \cdots \mathbf{A}_1$ while $\prod_{i=1}^{l} \mathbf{A}_i = \mathbf{A}_1 \mathbf{A}_2 \cdots \mathbf{A}_l$.

We now show that the link between the WMMSE and maximal MI objectives in a single-hop MIMO system established in [16] can be extended to multiuser multihop MIMO relay systems.

Theorem 1: By introducing a Hermitian weight matrix $\mathbf{W}$, the problem (10)–(12) has the same first order optimality condition as the following problem

$$\min_{\{\mathbf{B}_i\}, \{\mathbf{F}_l\}, \mathbf{W}} \quad tr\left(\mathbf{W}(\mathbf{I}_{N_0} + \mathbf{A}_L^H \mathbf{C}_L^{-1} \mathbf{A}_L)^{-1}\right) - \log|\mathbf{W}| \tag{14}$$
$$\text{s.t.} \quad tr\left(\mathbf{F}_l \mathbf{D}_{l-1} \mathbf{F}_l^H\right) \leq p_l, \quad l = 2, \ldots, L \tag{15}$$
$$tr\left(\mathbf{B}_i \mathbf{B}_i^H\right) \leq q_i, \quad i = 1, \ldots, N_u \tag{16}$$

when

$$\mathbf{W} = \mathbf{E}_L^{-1}. \tag{17}$$

Moreover, with given $\{\mathbf{B}_i\}$ and $\{\mathbf{F}_l\}$, the weight matrix $\mathbf{W}$ minimizing (14) is given by (17).

Proof: See Appendix A. □

Based on Theorem 1, we propose an iterative algorithm for the problem (10)–(12), where in each iteration, with $\mathbf{W}$ from the previous iteration, we first optimize $\{\mathbf{B}_i\}$ and $\{\mathbf{F}_l\}$ through solving the WMMSE problem (14)-(16). Then, we update $\mathbf{W}$ as (17) using $\{\mathbf{B}_i\}$ and $\{\mathbf{F}_l\}$ obtained in the current iteration. Note that the conditional updates of $\{\mathbf{B}_i\}$, $\{\mathbf{F}_l\}$ and $\mathbf{W}$ may either decrease or maintain but cannot increase the objective function (14). Monotonic convergence of the iterative algorithm towards (at least) a stationary point follows directly from this observation.

A. Decomposition of the WMMSE Matrix

With fixed $\mathbf{W}$, the second term in (14) is constant. Thus, the problem (14)–(16) can be rewritten as the following WMMSE problem

$$\min_{\{\mathbf{B}_i\}, \{\mathbf{F}_l\}} \quad tr\left(\mathbf{W}^{\frac{H}{2}} \mathbf{E}_L \mathbf{W}^{\frac{1}{2}}\right) \tag{18}$$
$$\text{s.t.} \quad tr\left(\mathbf{F}_l \mathbf{D}_{l-1} \mathbf{F}_l^H\right) \leq p_l, \quad l = 2, \ldots, L \tag{19}$$
$$tr\left(\mathbf{B}_i \mathbf{B}_i^H\right) \leq q_i, \quad i = 1, \ldots, N_u \tag{20}$$

where $\mathbf{W} = \mathbf{W}^{\frac{1}{2}} \mathbf{W}^{\frac{H}{2}}$ and $\mathbf{W}^{\frac{1}{2}} = \mathbf{W}^{\frac{H}{2}}$. The problem (18)–(20) is non-convex with matrix variables. A globally optimal solution is very difficult to obtain with reasonable computational complexity. However, the WMMSE matrix $\tilde{\mathbf{E}}_L \triangleq \mathbf{W}^{\frac{H}{2}} \mathbf{E}_L \mathbf{W}^{\frac{1}{2}}$ can be decomposed into L MMSE matrices as shown below.

Theorem 2: By introducing $N_{l-1} \times N_0$ matrices $\mathbf{T}_l$, $l = 2, \ldots, L$, the optimal $\{\mathbf{F}_l\}$ as the solution to the problem (18)–(20) can be written as

$$\mathbf{F}_l = \mathbf{T}_l \mathbf{W}^{\frac{H}{2}} \mathbf{A}_{l-1}^H \mathbf{D}_{l-1}^{-1}, \quad l = 2, \ldots, L. \tag{21}$$

With (21), $\tilde{\mathbf{E}}_L$ can be decomposed to

$$\tilde{\mathbf{E}}_L = \mathbf{W}^{\frac{H}{2}} \left(\mathbf{I}_{N_0} + \mathbf{F}_1^H \mathbf{H}_1^H \mathbf{H}_1 \mathbf{F}_1\right)^{-1} \mathbf{W}^{\frac{1}{2}} + \sum_{l=2}^{L} \left(\mathbf{T}_l^H \mathbf{H}_l^H \mathbf{H}_l \mathbf{T}_l + \mathbf{R}_l^{-1}\right)^{-1} \tag{22}$$

where

$$\mathbf{R}_l \triangleq \mathbf{W}^{\frac{H}{2}} \mathbf{A}_{l-1}^H \mathbf{D}_{l-1}^{-1} \mathbf{A}_{l-1} \mathbf{W}^{\frac{1}{2}}, \quad l = 2, \ldots, L. \tag{23}$$

In (21), $\{\mathbf{T}_l\} \triangleq \{\mathbf{T}_2, \ldots, \mathbf{T}_L\}$ is the optimal solution to the following problem

$$\min_{\{\mathbf{B}_i\},\{\mathbf{T}_l\}} \quad tr\left(\mathbf{W}^{\frac{H}{2}}(\mathbf{I}_{N_0}+\mathbf{F}_1^H\mathbf{H}_1^H\mathbf{H}_1\mathbf{F}_1)^{-1}\mathbf{W}^{\frac{1}{2}} + \sum_{l=2}^{L}\left(\mathbf{T}_l^H\mathbf{H}_l^H\mathbf{H}_l\mathbf{T}_l+\mathbf{R}_l^{-1}\right)^{-1}\right) \tag{24}$$

$$\text{s.t.} \quad tr(\mathbf{T}_l\mathbf{R}_l\mathbf{T}_l^H) \leq p_l, \quad l = 2,\ldots,L \tag{25}$$

$$tr(\mathbf{B}_i\mathbf{B}_i^H) \leq q_i, \quad i = 1,\ldots,N_u. \tag{26}$$

Proof: See Appendix B. □

We would like to note that the MMSE matrix decomposition for multihop MIMO relay systems has been discovered in [15] when $\mathbf{W}$ is an identity matrix. Therefore, Theorem 2 extends the result in [15] to the general case of $\mathbf{W} \neq \mathbf{I}_{N_0}$.

Using the matrix inversion lemma

$$(\mathbf{A}+\mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{D}\mathbf{A}^{-1}\mathbf{B}+\mathbf{C}^{-1})^{-1}\mathbf{D}\mathbf{A}^{-1} \tag{27}$$

we can rewrite $\mathbf{R}_l$, $l = 2,\ldots,L$, as

$$\mathbf{R}_l = \mathbf{W}^{\frac{H}{2}}\mathbf{A}_{l-1}^H\mathbf{C}_{l-1}^{-1}\mathbf{A}_{l-1}\left(\mathbf{A}_{l-1}^H\mathbf{C}_{l-1}^{-1}\mathbf{A}_{l-1}+\mathbf{I}_{N_0}\right)^{-1}\mathbf{W}^{\frac{1}{2}}.$$

In the case of (moderately) high SNR where $\mathbf{A}_{l-1}^H\mathbf{C}_{l-1}^{-1}\mathbf{A}_{l-1} \gg \mathbf{I}_{N_0}$, $l = 2,\ldots,L$, we have $\mathbf{R}_l \approx \mathbf{W}$, $l = 2,\ldots,L$. This indicates that in this case, $\{\mathbf{B}_i\}$ and $\{\mathbf{T}_l\}$ have almost no impact on $\mathbf{R}_l$, $l = 2,\ldots,L$, which implies that the objective function (24) and the constraints in (25) are decoupled with respect to the variables $\{\mathbf{B}_i\}$ and $\{\mathbf{T}_l\}$. Thus, the problem (24)–(26) can be approximated and decomposed into the source precoding matrices optimization problem

$$\min_{\{\mathbf{B}_i\}} tr\left(\mathbf{W}^{\frac{H}{2}}(\mathbf{I}_{N_0}+\mathbf{F}_1^H\mathbf{H}_1^H\mathbf{H}_1\mathbf{F}_1)^{-1}\mathbf{W}^{\frac{1}{2}}\right) \tag{28}$$

$$\text{s.t.} \quad tr(\mathbf{B}_i\mathbf{B}_i^H) \leq q_i, \quad i = 1,\ldots,N_u \tag{29}$$

and the relay precoding matrix optimization problem for each $\mathbf{T}_l$, $l = 2,\ldots,L$

$$\min_{\mathbf{T}_l} tr\left(\left(\mathbf{T}_l^H\mathbf{H}_l^H\mathbf{H}_l\mathbf{T}_l+\mathbf{R}_l^{-1}\right)^{-1}\right) \tag{30}$$

$$\text{s.t.} \quad tr\left(\mathbf{T}_l\mathbf{R}_l\mathbf{T}_l^H\right) \leq p_l. \tag{31}$$

In the next two subsections, we focus on solving the problem (28)–(29) and the problem (30)–(31).

B. The Source Matrices Optimization

When $\mathbf{W} = \mathbf{I}_{N_0}$, it is shown in [15] that the problem (28)–(29) can be converted to a convex semidefinite programming (SDP) problem. However, for general $\mathbf{W} \neq \mathbf{I}_{N_0}$, the problem (28)–(29) cannot be cast as a convex optimization problem. Interestingly, as (28) is the WMMSE of the single-hop multiuser MIMO system (2), it can be written as

$$tr\left(\mathbf{W}^{\frac{H}{2}}(\mathbf{I}_{N_0}+\mathbf{F}_1^H\mathbf{H}_1^H\mathbf{H}_1\mathbf{F}_1)^{-1}\mathbf{W}^{\frac{1}{2}}\right) = \min_{\mathbf{L}} tr(\mathbf{W}E[(\mathbf{L}^H\mathbf{y}_1-\mathbf{s})(\mathbf{L}^H\mathbf{y}_1-\mathbf{s})^H]) \tag{32}$$

where $\mathbf{L}$ is the weight matrix of the linear receiver for the MIMO system in (2). To see this, let us work out the expectation on the right-hand side of (32) as

$$\begin{aligned} &tr(\mathbf{W}E[(\mathbf{L}^H\mathbf{y}_1-\mathbf{s})(\mathbf{L}^H\mathbf{y}_1-\mathbf{s})^H]) \\ &\quad = tr(\mathbf{W}[\mathbf{L}^H(\mathbf{H}_1\mathbf{F}_1\mathbf{F}_1^H\mathbf{H}_1^H+\mathbf{I}_{N_1})\mathbf{L} \\ &\qquad - \mathbf{L}^H\mathbf{H}_1\mathbf{F}_1 - \mathbf{F}_1^H\mathbf{H}_1^H\mathbf{L}+\mathbf{I}_{N_0}]). \end{aligned} \tag{33}$$

The optimal $\mathbf{L}$ minimizing (33) is the Wiener filter [25] given by

$$\mathbf{L} = (\mathbf{H}_1\mathbf{F}_1\mathbf{F}_1^H\mathbf{H}_1^H+\mathbf{I}_{N_1})^{-1}\mathbf{H}_1\mathbf{F}_1. \tag{34}$$

By substituting (34) back to (33), we obtain the left-hand side of (32).

By exploiting (32), the problem (28)–(29) can be solved via the following problem

$$\min_{\{\mathbf{B}_i\},\mathbf{L}} \quad tr\left((\mathbf{W}^{\frac{H}{2}}\mathbf{L}^H\mathbf{H}_1\mathbf{F}_1-\mathbf{W}^{\frac{H}{2}})(\mathbf{W}^{\frac{H}{2}}\mathbf{L}^H\mathbf{H}_1\mathbf{F}_1-\mathbf{W}^{\frac{H}{2}})^H + \mathbf{W}^{\frac{H}{2}}\mathbf{L}^H\mathbf{L}\mathbf{W}^{\frac{1}{2}}\right) \tag{35}$$

$$\text{s.t.} \quad tr\left(\mathbf{B}_i\mathbf{B}_i^H\right) \leq q_i, \quad i = 1,\ldots,N_u. \tag{36}$$

In the following, we propose an iterative algorithm for the problem (35)–(36). In each iteration, we first optimize $\mathbf{L}$ as given by (34) based on $\{\mathbf{B}_i\}$ from the previous iteration. Then using $\mathbf{L}$ obtained in the current iteration, we optimize $\{\mathbf{B}_i\}$ by solving the problem of

$$\min_{\{\mathbf{B}_i\}} tr\left(\left(\mathbf{Z}\mathbf{F}_1-\mathbf{W}^{\frac{H}{2}}\right)\left(\mathbf{Z}\mathbf{F}_1-\mathbf{W}^{\frac{H}{2}}\right)^H\right) \tag{37}$$

$$\text{s.t.} \quad tr\left(\mathbf{B}_i\mathbf{B}_i^H\right) \leq q_i, \quad i = 1,\ldots,N_u \tag{38}$$

where $\mathbf{Z} \triangleq \mathbf{W}^{\frac{H}{2}}\mathbf{L}^H\mathbf{H}_1$. We update $\{\mathbf{B}_i\}$ and $\mathbf{L}$ alternatingly till convergence.

Let us introduce $\mathbf{Z}_i$ and $\mathbf{W}_i$ which contain the $\sum_{j=0}^{i-1} M_j+1$ to $\sum_{j=0}^{i} M_j$ columns of $\mathbf{Z}$ and $\mathbf{W}^{\frac{H}{2}}$ respectively, $i = 1,\ldots,N_u$, where $M_0 = 0$. We can rewrite (37) as

$$\sum_{i=1}^{N_u} tr((\mathbf{Z}_i\mathbf{B}_i-\mathbf{W}_i)(\mathbf{Z}_i\mathbf{B}_i-\mathbf{W}_i)^H). \tag{39}$$

It can be seen from (38) and (39) that the problem (37)–(38) can be decomposed into N_u subproblems, where each $\mathbf{B}_i$ is optimized through solving the following problem

$$\min_{\mathbf{B}_i} tr((\mathbf{Z}_i\mathbf{B}_i-\mathbf{W}_i)(\mathbf{Z}_i\mathbf{B}_i-\mathbf{W}_i)^H) \tag{40}$$

$$\text{s.t.} \quad tr\left(\mathbf{B}_i\mathbf{B}_i^H\right) \leq q_i. \tag{41}$$

Using the Lagrange multiplier method [26], the solution to the problem (40)–(41) is given by

$$\mathbf{B}_i = \left(\mathbf{Z}_i^H\mathbf{Z}_i+\lambda_i\mathbf{I}_{M_i}\right)^{-1}\mathbf{Z}_i^H\mathbf{W}_i, \quad i = 1,\ldots,N_u \tag{42}$$

where $\lambda_i \geq 0$ is the Lagrangian multiplier and can be found by substituting (42) back into (41) and solve the obtained equation using the bisection search [26].

We would like to mention that the conditional updates of $\{\mathbf{B}_i\}$ and $\mathbf{L}$ may either decrease or maintain but cannot increase the objective function (35). Monotonic convergence of

TABLE I
PROCEDURE OF THE PROPOSED SOURCE AND RELAY MATRICES DESIGN ALGORITHM

1) Initialize the algorithm with $\mathbf{W}^{(0)} = \mathbf{I}_{N_0}$, $\mathbf{B}_i^{(0)} = \sqrt{q_i/M_i}\,\mathbf{I}_{M_i}$, $i = 1, \cdots, N_u$, and $\mathbf{F}_l^{(0)} = \sqrt{p_l/tr(\mathbf{D}_{l-1}^{(0)})}\,\mathbf{I}_{N_{l-1}}$, $l = 2, \cdots, L$; Set $n = 0$.
2) Set $m = 0$, $\{\mathbf{B}_i^{[0]}\} = \{\mathbf{B}_i^{(n)}\}$, and
   a) Update $\mathbf{L}^{[m]}$ as (34) with fixed $\{\mathbf{B}_i^{[m]}\}$.
   b) For $i = 1, \cdots, N_u$, update $\mathbf{B}_i^{[m+1]}$ as (42) by solving the problem (40)-(41) with fixed $\mathbf{L}^{[m]}$ and $\mathbf{W}^{(n)}$.
   c) If $\max\left|\{\mathbf{B}_i^{[m+1]}\} - \{\mathbf{B}_i^{[m]}\}\right|_1 \leq \varepsilon_1$, then $\{\mathbf{B}_i^{(n+1)}\} = \{\mathbf{B}_i^{[m+1]}\}$; end of step 2.
   Otherwise, let $m := m + 1$ and go to Step 2a.
3) For $l = 2, \cdots, L$, update $\mathbf{F}_l^{(n+1)}$ as (44) by solving the problem (45)-(46) with fixed $\{\mathbf{B}_i^{(n+1)}\}$ and $\mathbf{W}^{(n)}$.
4) If $\max\left|\{\mathbf{F}_l^{(n+1)}\} - \{\mathbf{F}_l^{(n)}\}\right|_1 \leq \varepsilon_2$, then end.
   Otherwise, update $\mathbf{W}^{(n+1)}$ as (17) with given $\{\mathbf{F}_l^{(n+1)}\}$ and $\{\mathbf{B}_i^{(n+1)}\}$; let $n := n + 1$ and go to Step 2.

the source matrices optimization algorithm towards (at least) a stationary point follows directly from this observation.

C. The Relay Matrices Optimization

Let us introduce the eigenvalue decomposition (EVD) of $\mathbf{H}_l^H\mathbf{H}_l = \mathbf{V}_l\mathbf{\Lambda}_l\mathbf{V}_l^H$ and $\mathbf{R}_l = \mathbf{U}_l\mathbf{\Sigma}_l\mathbf{U}_l^H$, $l = 2, \ldots, L$, where $\mathbf{\Lambda}_l$ and $\mathbf{V}_l$ are $N_{l-1} \times N_{l-1}$ matrices, the dimensions of $\mathbf{U}_l$ and $\mathbf{\Sigma}_l$ are $N_0 \times N_0$, and the diagonal elements of $\mathbf{\Lambda}_l$ and $\mathbf{\Sigma}_l$ are both sorted in descending order. It can be shown using Lemma 2 in [15] that the solution to the relay matrices optimization problem (30)–(31) has a water-filling solution as

$$\mathbf{T}_l = \mathbf{V}_{l,1}\mathbf{\Delta}_l\mathbf{U}_l^H, \quad l = 2, \ldots, L \tag{43}$$

where $\mathbf{V}_{l,1}$ denotes the leftmost N_0 columns of $\mathbf{V}_l$, and $\mathbf{\Delta}_l$ is an $N_0 \times N_0$ diagonal matrix that remains to be optimized. Based on (21) and (43), the relay matrices are given by

$$\mathbf{F}_l = \mathbf{V}_{l,1}\mathbf{\Delta}_l{\mathbf{U}_l}^H\mathbf{W}^{\frac{H}{2}}\mathbf{A}_{l-1}^H\mathbf{D}_{l-1}^{-1}, \quad l = 2, \ldots, L. \tag{44}$$

Substituting (43) back into (30)–(31), we obtain the following optimal power loading problem with scalar variables

$$\min_{\delta_{l,1},\ldots,\delta_{l,N_0}} \sum_{i=1}^{N_0} \frac{1}{\sigma_{l,i}^{-1} + \delta_{l,i}^2\lambda_{l,i}} \tag{45}$$

$$\text{s.t.} \sum_{i=1}^{N_0} \delta_{l,i}^2\sigma_{l,i} \leq p_l \tag{46}$$

where $\delta_{l,i}, \sigma_{l,i}, \lambda_{l,i}, i = 1, \ldots, N_0$, denote the ith diagonal element of $\mathbf{\Delta}_l, \mathbf{\Sigma}_l, \mathbf{\Lambda}_l$, respectively. The problem (45)–(46) can be solved by the Lagrange multiplier method as

$$\delta_{l,i}^2 = \frac{1}{\lambda_{l,i}}\left(\sqrt{\frac{\lambda_{l,i}}{\mu_l\sigma_{l,i}}} - \frac{1}{\sigma_{l,i}}\right)^+, i = 1, \ldots, N_0 \tag{47}$$

where $(x)^+ \triangleq \max(x, 0)$, and $\mu_l > 0$ is the Lagrangian multiplier and the solution to the nonlinear equation of $\sum_{i=1}^{N_0} \frac{\sigma_{l,i}}{\lambda_{l,i}}(\sqrt{\frac{\lambda_{l,i}}{\mu_l\sigma_{l,i}}} - \frac{1}{\sigma_{l,i}})^+ = p_l$.

D. Summary and Comments

The procedure of the proposed source and relay matrices design algorithm is summarized in Table I, where ε_1 and ε_2 are small positive numbers close to zero up to which convergence is acceptable, $\max|\cdot|_1$ stands for the maximum of the absolute value of all elements in a matrix, and the superscript (n) and $[m]$ denote the number of iterations at the outer loop and the inner loop, respectively[2].

The major operation in each iteration of the proposed algorithm involves matrix inversion and matrix EVD. Thus, the per-iteration computational complexity order of the proposed algorithm is $\mathcal{O}(\sum_{l=0}^{L-1} N_l^3)$. The overall complexity depends on the number of iterations till convergence. It will be shown in Section IV that the proposed algorithm converges usually in less than 10 iterations.

Interestingly, as $\mathbf{R}_l$ can be approximated as $\mathbf{W}$ at (moderately) high SNRs, the relay matrices optimization problem can be further simplified by substituting $\mathbf{R}_l$ in (30)–(31) with $\mathbf{W}$, which can be rewritten as

$$\min_{\mathbf{T}_l} tr\left(\left(\mathbf{T}_l^H\mathbf{H}_l^H\mathbf{H}_l\mathbf{T}_l + \mathbf{W}^{-1}\right)^{-1}\right) \tag{48}$$

$$\text{s.t. } tr(\mathbf{T}_l\mathbf{W}\mathbf{T}_l^H) \leq p_l. \tag{49}$$

By introducing the EVD of $\mathbf{W} = \mathbf{U}_w\mathbf{\Sigma}\mathbf{U}_w^H$ and using Lemma 2 in [15], the solution to the problem (48)–(49) is given by

$$\mathbf{T}_l = \mathbf{V}_{l,1}\mathbf{\Theta}_l\mathbf{U}_w^H, \quad l = 2, \ldots, L. \tag{50}$$

Based on (21) and (50), the relay matrices are given by

$$\mathbf{F}_l = \mathbf{V}_{l,1}\mathbf{\Theta}_l\mathbf{U}_w^H\mathbf{W}^{\frac{H}{2}}\mathbf{A}_{l-1}^H\mathbf{D}_{l-1}^{-1}, \quad l = 2, \ldots, L. \tag{51}$$

The diagonal elements of $\mathbf{\Theta}_l$ are given by

$$\theta_{l,i}^2 = \frac{1}{\lambda_{l,i}}\left(\sqrt{\frac{\lambda_{l,i}}{\nu_l\sigma_i}} - \frac{1}{\sigma_i}\right)^+, \quad i = 1, \ldots, N_0 \tag{52}$$

where σ_i, $i = 1, \ldots, N_0$, denotes the ith diagonal element of $\mathbf{\Sigma}$. The Lagrangian multiplier $\nu_l > 0$ is determined by $\sum_{i=1}^{N_0} \frac{\sigma_i}{\lambda_{l,i}}(\sqrt{\frac{\lambda_{l,i}}{\nu_l\sigma_i}} - \frac{1}{\sigma_i})^+ = p_l$.

Obviously, $\mathbf{R}_l$ does not need to be calculated in the problem (48)–(49). Thus, the simplified relay design has a lower computational complexity than the algorithm which solves the problem (30)–(31). To apply the simplified relay design, we only need to change Step 3 in Table I to update $\mathbf{F}_l^{(n+1)}$ as (51) with fixed $\{\mathbf{B}_i^{(n+1)}\}$ and $\mathbf{W}^{(n)}$. It will be shown in the next section that for two-hop relay systems, the MI performance of this simplified relay design is slightly worse than that of the algorithm solving the problem (30)–(31).

IV. NUMERICAL EXAMPLES

In this section, we study the performance of the proposed source and relay precoding matrices design algorithms through

[2]We have also tried the beamforming-based initialization, where $\mathbf{B}_i = \sqrt{q_i/M_i}\mathbf{V}_{G_i}$, $i = 1, \ldots, N_u$ and $\mathbf{F}_l = \sqrt{p_l/tr(\mathbf{D}_{l-1})}\mathbf{V}_{H_l}$, $l = 2, \ldots, L$. Here $\mathbf{V}_{G_i}$ and $\mathbf{V}_{H_l}$ are the right singular vector matrix of $\mathbf{G}_i$ and $\mathbf{H}_l$, respectively. We observed that the beamforming-based initialization results in almost identical MI performance as the initialization used here.

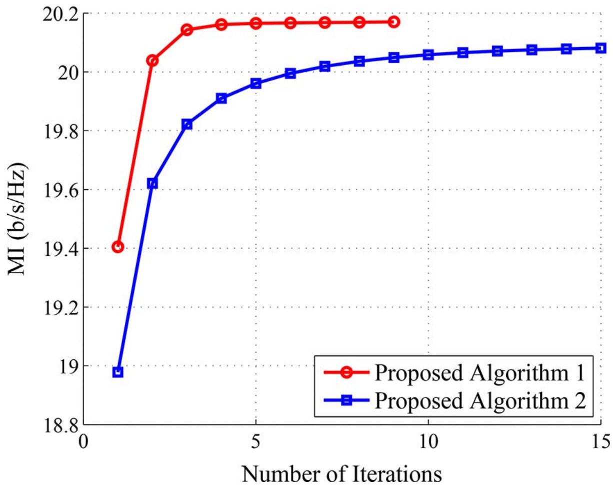

Fig. 2. Example 1: MI versus the number of iterations. $L = 2$, $N_u = 2$, $M_1 = 3$, $M_2 = 2$, $N_1 = N_2 = 6$, and $P = Q = 20$ dB.

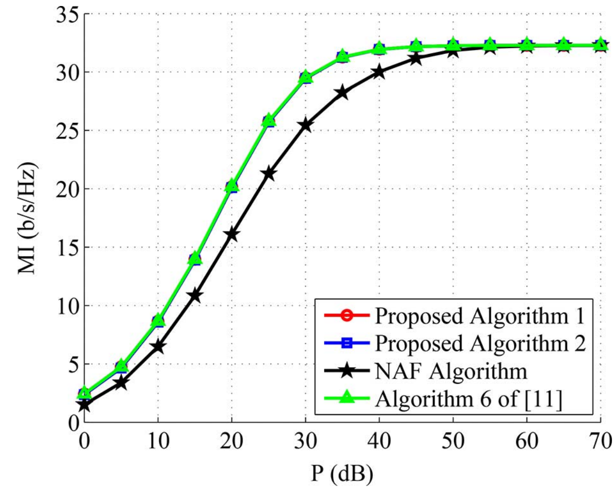

Fig. 3. Example 1: MI versus P. $L = 2$, $N_u = 2$, $M_1 = 3$, $M_2 = 2$, $N_1 = N_2 = 6$, and $Q = 20$ dB.

numerical simulations. We simulate a flat Rayleigh fading environment where all channel matrices have entries with zero mean. To normalize the effect of the number of transmit antennas to the SNR, the variance of entries in $\mathbf{G}_i$ is set to be $1/M_i$, $i = 1, \ldots, N_u$, and the variance of entries in $\mathbf{H}_l$ is set to be $1/N_{l-1}, l = 2, \ldots, L$. All noises are complex circularly symmetric with zero mean and unit variance. In order to study the system MI versus the power constraint at the source and relay nodes, we assume that all relay nodes have the same transmission power constraint P, i.e., $p_l = P$, $l = 2, \ldots, L$, and each user's transmission power budget is Q, i.e., $q_i = Q$, $i = 1, \ldots, N_u$. The variables ε_1 and ε_2 for stopping the iterations in the proposed algorithm are both set to be 10^{-3}.

We compare the performance of the proposed algorithm described in Table I (denoted as Proposed Algorithm 1), the proposed algorithm with the simplified relay matrices design in (51) (denoted as Proposed Algorithm 2), and the naive amplify-and-forward (NAF) algorithm where all source and relay precoding matrices are scaled identity matrices satisfying the power constrains. In addition, for simulation examples with two-hop relay systems, we also compare with the Algorithm 6 proposed in [11].

In the first example, we simulate a two-hop ($L = 2$) MIMO relay system with $N_u = 2$, $M_1 = 3$, $M_2 = 2$, $N_1 = N_2 = 6$, and $P = Q = 20$ dB. The MI from the proposed algorithms at different number of iterations ($\{\mathbf{B}_i\}$, $\{\mathbf{F}_l\}$, and $\mathbf{W}$ are updated in each iteration) is shown in Fig. 2. It can be clearly seen that the MI from both algorithms increases with iterations and only a few iterations are required for both algorithms to converge. In fact, we observed in many simulations that the Proposed Algorithms 1 and 2 converge in 4–5 and 10–12 iterations, respectively. It can also be seen in Fig. 2 that although the Proposed Algorithm 2 saves per-iteration computational complexity by approximating $\mathbf{R}_l$ as $\mathbf{W}$, the Proposed Algorithm 1 has faster convergence speed and a better MI performance than the Proposed Algorithm 2.

The MI performance of all algorithms tested versus P is shown in Fig. 3 with $Q = 20$ dB. It can be seen that the system MI yielded by four algorithms increases as P increases. Both proposed algorithms and the Algorithm 6 of [11] outperform the NAF algorithm in terms of the system MI. In this scenario, the proposed two algorithms yield almost the same MI as the Algorithm 6 of [11]. The MI performance of the Proposed Algorithm 1 is only slightly better than that of the Proposed Algorithm 2. Moreover, we observe from Fig. 3 that at high P level (above 47 dB), the increasing of the system MI versus P is very marginal. The reason is that the performance of a MIMO relay system is subjected to both source power constraint Q and relay power constraint P. When Q is fixed, the performance of all algorithms has the saturation effect as P increases.

The MI performance of all algorithms versus Q is demonstrated in Fig. 4 with P fixed at 20 dB. Similar to Fig. 3, we observe from Fig. 4 that the system MI by all algorithms improves as Q increases. The proposed algorithms and the Algorithm 6 in [11] yield higher system MI than the NAF algorithm. Similar to Fig. 3, the proposed algorithms have almost the same MI performance as the Algorithm 6 in [11]. In particular, the Proposed Algorithm 1 has slightly higher MI than the Algorithm 6 in [11] at low Q level. Thus, both proposed algorithms in this paper are efficient in optimizing the system MI in a two-hop multiuser MIMO relay system.

In the second example, we simulate a four-hop ($L = 4$) MIMO relay system with $N_u = 3$ users to demonstrate that the proposed algorithms can be extended to multihop multiuser systems. For the sake of notational simplicity, we assume that all users have the same number of antennas with $M_i = M = 3$, $i = 1, \ldots, N_u$, and all relay nodes have the same number of antennas, i.e., $N_l = N = 8$, $l = 1, \ldots, L-1$. Fig. 5 shows the system MI of the proposed algorithms at different number of iterations with $P = Q = 20$ dB. Comparing Fig. 5 with Fig. 2, it can be seen that two proposed algorithms have similar convergence behavior in four-hop and two-hop MIMO relay systems.

The MI performance of three algorithms in the four-hop relay system versus P at $Q = 20$dB is shown in Fig. 6. Different to Fig. 3, it can be seen from Fig. 6 that at low P level (below 25 dB), there is apparent difference between MI of the two proposed algorithms. However, as P increases, the MI gap of two algorithms reduces. This is because the condition of approximating $\mathbf{R}_l$ as $\mathbf{W}$ is in high SNR scenarios. As P increases, $\mathbf{R}_l$

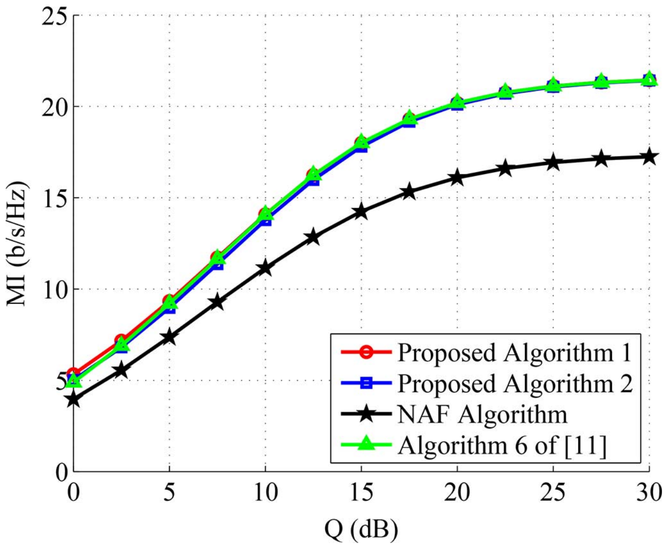

Fig. 4. Example 1: MI versus Q. $L = 2$, $N_u = 2$, $M_1 = 3$, $M_2 = 2$, $N_1 = N_2 = 6$, and $P = 20$ dB.

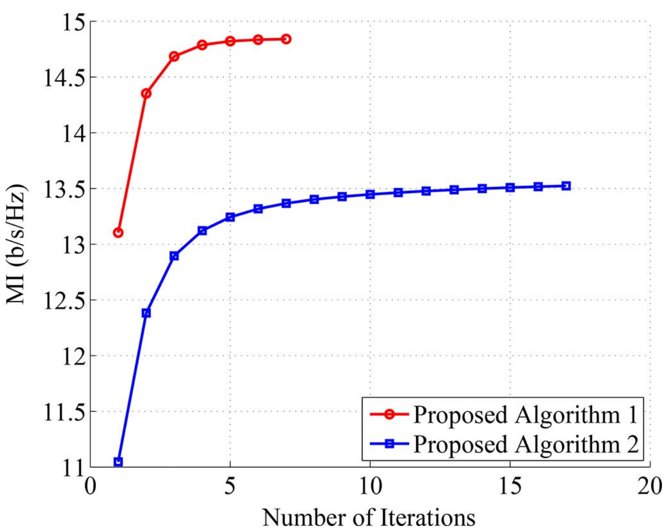

Fig. 5. Example 2: MI versus the number of iterations. $L = 4$, $N_u = 3$, $M = 3$, $N = 8$, and $P = Q = 20$ dB.

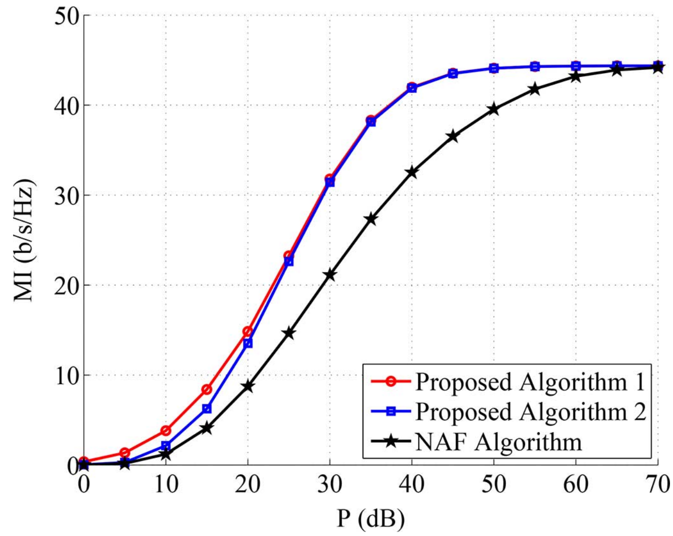

Fig. 6. Example 2: MI versus P. $L = 4$, $N_u = 3$, $M = 3$, $N = 8$, and $Q = 20$ dB.

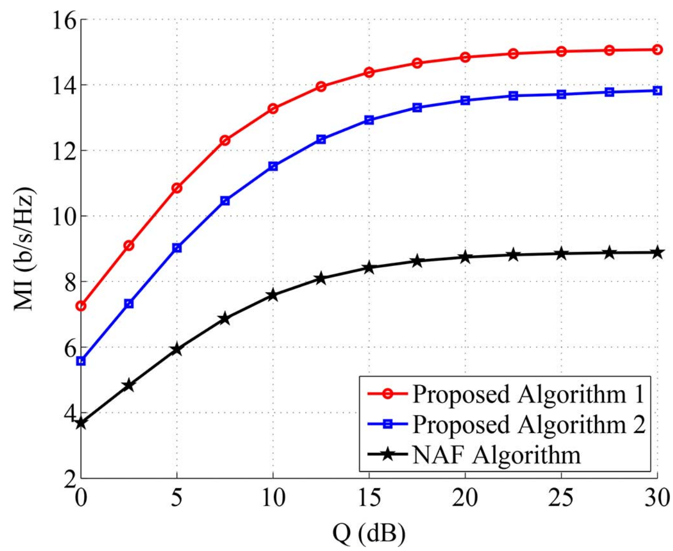

Fig. 7. Example 2: MI versus Q. $L = 4$, $N_u = 3$, $M = 3$, $N = 8$, and $P = 20$ dB.

is getting closer to $\mathbf{W}$, and thus, the MI gap between two proposed algorithms becomes smaller.

Finally, Fig. 7 illustrates the MI performance of three algorithms versus Q with P fixed at 20 dB. It is clear from Fig. 7 that both proposed algorithms have much higher MI than the NAF algorithm. Moreover, different from the two-hop system, the MI performance of Algorithm 1 is obviously better than that of the Algorithm 2 over the whole range of Q in the four-hop system. Considering the convergence properties and the MI performance, Algorithm 1 is more suitable for multihop (especially $L \geq 4$) multiuser MIMO relay systems.

V. Conclusion

We have proposed source and relay precoding matrices design algorithms for a multiuser multihop MIMO relay system. By exploiting the link between the maximal MI and the WMMSE objectives, an iterative algorithm has been developed to maximize the system MI by solving the WMMSE problem at each iteration. It has been shown that the WMMSE matrix of the signal waveform estimation at the destination node can be decomposed into the sum of the WMMSE matrices at all relay nodes, which greatly reduces the computational complexity at a (moderately) high SNR environment. Numerical examples have shown the effectiveness of the proposed algorithms.

Appendix A
Proof of Theorem 1

Using (13), the objective function (10) can be rewritten as

$$\min_{\{\mathbf{B}_i\},\{\mathbf{F}_l\}} \log|\mathbf{E}_L|. \tag{53}$$

The objective function (14) can be equivalently rewritten as

$$\min_{\{\mathbf{B}_i\},\{\mathbf{F}_l\},\mathbf{W}} tr(\mathbf{W}\mathbf{E}_L) - \log|\mathbf{W}|. \tag{54}$$

Based on the chain rule of matrix derivatives ([24], (137)), the derivative of (53) with respect to $\mathbf{B}_i$ or $\mathbf{F}_l$ is given by

$$\frac{\partial \log|\mathbf{E}_L|}{\partial \mathbf{M}} = tr\left(\left(\frac{\partial \log|\mathbf{E}_L|}{\partial \mathbf{E}_L}\right)^T \frac{\partial \mathbf{E}_L}{\partial \mathbf{M}}\right) = tr\left(\mathbf{E}_L^{-1}\frac{\partial \mathbf{E}_L}{\partial \mathbf{M}}\right) \quad (55)$$

where $\mathbf{M}$ can be either $\mathbf{B}_i$ or $\mathbf{F}_l$ and the identity of $\partial \log|\mathbf{X}|/\partial \mathbf{X} = (\mathbf{X}^{-1})^T$ ([24], (57)) is used.

Similarly, by using the chain rule of matrix derivatives and the identity of $\partial\, tr(\mathbf{AX})/\partial \mathbf{X} = \mathbf{A}^T$ ([24], (100)), we obtain the derivative of (54) with respect to $\mathbf{B}_i$ or $\mathbf{F}_l$ as

$$\frac{\partial tr(\mathbf{W}\mathbf{E}_L)}{\partial \mathbf{M}} = tr\left(\left(\frac{\partial tr(\mathbf{W}\mathbf{E}_L)}{\partial \mathbf{E}_L}\right)^T \frac{\partial \mathbf{E}_L}{\partial \mathbf{M}}\right) = tr\left(\mathbf{W}\frac{\partial \mathbf{E}_L}{\partial \mathbf{M}}\right). \quad (56)$$

It can be clearly seen that (56) equals to (55) when (17) holds. This shows that under (17), the problem (10)–(12) has the same first order optimality condition as the problem (14)–(16).

The derivative of (54) with respect to $\mathbf{W}$ can be written as

$$\frac{\partial tr(\mathbf{W}\mathbf{E}_L) - \log|\mathbf{W}|}{\partial \mathbf{W}} = \mathbf{E}_L^T - (\mathbf{W}^{-1})^T \quad (57)$$

By equating (57) to zero, we obtain (17). Thus with given $\{\mathbf{B}_i\}$ and $\{\mathbf{F}_l\}$, the weight matrix $\mathbf{W}$ minimizing (54) is given by (17).

Appendix B
Proof of Theorem 2

The WMMSE matrix $\tilde{\mathbf{E}}_L$ can be rewritten as

$$\begin{aligned}\tilde{\mathbf{E}}_L &= \mathbf{W}^{\frac{H}{2}}\left(\mathbf{I}_{N_0} - \mathbf{A}_{L-1}^H\mathbf{F}_L^H\mathbf{H}_L^H\left(\mathbf{H}_L\mathbf{F}_L\mathbf{D}_{L-1}\mathbf{F}_L^H\mathbf{H}_L^H\right.\right.\\ &\quad \left.\left.+\mathbf{I}_{N_L}\right)^{-1}\mathbf{H}_L\mathbf{F}_L\mathbf{A}_{L-1}\right)\mathbf{W}^{\frac{1}{2}} \quad (58)\\ &= \mathbf{W}^{\frac{H}{2}}\left(\mathbf{I}_{N_0} - \mathbf{A}_{L-1}^H\left(\mathbf{D}_{L-1}^{-1} - \left(\mathbf{D}_{L-1}\mathbf{F}_L^H\mathbf{H}_L^H\mathbf{H}_L\mathbf{F}_L\right.\right.\right.\\ &\quad \left.\left.\left.\times\mathbf{D}_{L-1} + \mathbf{D}_{L-1}\right)^{-1}\right)\mathbf{A}_{L-1}\right)\mathbf{W}^{\frac{1}{2}} \quad (59)\\ &= \mathbf{W}^{\frac{H}{2}}\left(\mathbf{I}_{N_0} + \mathbf{A}_{L-1}^H\mathbf{C}_{L-1}^{-1}\mathbf{A}_{L-1}\right)^{-1}\mathbf{W}^{\frac{1}{2}} + \tilde{\mathbf{A}}_{L-1}^H\\ &\quad \times\left(\mathbf{D}_{L-1}\mathbf{F}_L^H\mathbf{H}_L^H\mathbf{H}_L\mathbf{F}_L\mathbf{D}_{L-1} + \mathbf{D}_{L-1}\right)^{-1}\tilde{\mathbf{A}}_{L-1} \quad (60)\end{aligned}$$

where $\tilde{\mathbf{A}}_{L-1} \triangleq \mathbf{A}_{L-1}\mathbf{W}^{\frac{1}{2}}$. The matrix inversion lemma (27) is used to obtain (58) and (60), and the identity of $\mathbf{B}^H(\mathbf{BCB}^H + \mathbf{I})^{-1}\mathbf{B} = \mathbf{C}^{-1} - (\mathbf{CB}^H\mathbf{BC} + \mathbf{C})^{-1}$ is applied to get (59).

It can be seen that the first term in (60) is irrelevant to $\mathbf{F}_L$. Therefore, the problem of optimizing $\mathbf{F}_L$ can be written as

$$\min_{\mathbf{F}_L} tr\left(\tilde{\mathbf{A}}_{L-1}^H(\mathbf{D}_{L-1}\mathbf{F}_L^H\mathbf{H}_L^H\mathbf{H}_L\mathbf{F}_L\mathbf{D}_{L-1}+\mathbf{D}_{L-1})^{-1}\tilde{\mathbf{A}}_{L-1}\right) \quad (61)$$

$$\text{s.t. } tr(\mathbf{F}_L\mathbf{D}_{L-1}\mathbf{F}_L^H) \le p_L. \quad (62)$$

By introducing $\tilde{\mathbf{F}}_L = \mathbf{F}_L\mathbf{D}_{L-1}^{\frac{1}{2}}$, the problem (61)–(62) can be rewritten as

$$\min_{\mathbf{F}_L} tr\left(\mathbf{\Psi}_{L-1}^H(\tilde{\mathbf{F}}_L^H\mathbf{H}_L^H\mathbf{H}_L\tilde{\mathbf{F}}_L+\mathbf{I}_{L-1})^{-1}\mathbf{\Psi}_{L-1}\right) \quad (63)$$

$$\text{s.t. } tr(\tilde{\mathbf{F}}_L\tilde{\mathbf{F}}_L^H) \le p_L \quad (64)$$

where $\mathbf{\Psi}_{L-1} \triangleq \mathbf{D}_{L-1}^{-\frac{1}{2}}\mathbf{A}_{L-1}\mathbf{W}^{\frac{1}{2}}$.

Let us introduce the EVD of $\mathbf{H}_L^H\mathbf{H}_L = \mathbf{V}_L\mathbf{\Lambda}_L\mathbf{V}_L^H$, and the singularvalue decomposition (SVD) of $\mathbf{\Psi}_{L-1} = \mathbf{U}_\Psi\mathbf{\Sigma}_\Psi\mathbf{V}_\Psi^H$, where $\mathbf{\Lambda}_L$ and $\mathbf{V}_L$ are $N_{L-1}\times N_{L-1}$ matrices, the dimensions of $\mathbf{U}_\Psi$, $\mathbf{\Sigma}_\Psi$, $\mathbf{V}_\Psi$ are $N_{L-1}\times N_0$, $N_0\times N_0$, $N_0\times N_0$, respectively, and the diagonal elements of $\mathbf{\Lambda}_L$ and $\mathbf{\Sigma}_\Psi$ are both sorted in descending order. Based on Lemma 2 in [15], the SVD of the optimal $\tilde{\mathbf{F}}_L$ is given by $\tilde{\mathbf{F}}_L = \mathbf{V}_{L,1}\mathbf{\Omega}_L\mathbf{U}_\Psi^H$, where $\mathbf{\Omega}_L$ is the $N_0\times N_0$ diagonal singular value matrix, and $\mathbf{V}_{L,1}$ denotes the leftmost N_0 columns of $\mathbf{V}_L$. So we have

$$\tilde{\mathbf{F}}_L = \mathbf{V}_{L,1}\mathbf{\Omega}_L\mathbf{\Sigma}_\Psi^{-1}\mathbf{V}_\Psi^H\mathbf{V}_\Psi\mathbf{\Sigma}_\Psi\mathbf{U}_\Psi^H = \mathbf{T}_L\mathbf{\Psi}_{L-1}^H \quad (65)$$

where $\mathbf{T}_L \triangleq \mathbf{V}_{L,1}\mathbf{\Omega}_L\mathbf{\Sigma}_\Psi^{-1}\mathbf{V}_\Psi^H$, and

$$\mathbf{F}_L = \mathbf{T}_L\mathbf{W}^{\frac{H}{2}}\mathbf{A}_{L-1}^H\mathbf{D}_{L-1}^{-1}. \quad (66)$$

Using (66) and the matrix inversion lemma (27), the second term in (60) can be rewritten as

$$\begin{aligned}&\tilde{\mathbf{A}}_{L-1}^H\left(\tilde{\mathbf{A}}_{L-1}\mathbf{T}_L^H\mathbf{H}_L^H\mathbf{H}_L\mathbf{T}_L\tilde{\mathbf{A}}_{L-1}^H + \mathbf{D}_{L-1}\right)^{-1}\tilde{\mathbf{A}}_{L-1}\\ &= \tilde{\mathbf{A}}_{L-1}^H\left[\mathbf{D}_{L-1}^{-1} - \mathbf{D}_{L-1}^{-1}\tilde{\mathbf{A}}_{L-1}\left(\tilde{\mathbf{A}}_{L-1}^H\mathbf{D}_{L-1}^{-1}\tilde{\mathbf{A}}_{L-1}\right.\right.\\ &\quad \left.\left.+(\mathbf{T}_L^H\mathbf{H}_L^H\mathbf{H}_L\mathbf{T}_L)^{-1}\right)^{-1}\tilde{\mathbf{A}}_{L-1}^H\mathbf{D}_{L-1}^{-1}\right]\tilde{\mathbf{A}}_{L-1}\\ &= \left[\mathbf{T}_L^H\mathbf{H}_L^H\mathbf{H}_L\mathbf{T}_L + \left(\tilde{\mathbf{A}}_{L-1}^H\mathbf{D}_{L-1}^{-1}\tilde{\mathbf{A}}_{L-1}\right)^{-1}\right]^{-1}. \quad (67)\end{aligned}$$

Substituting (67) back into (60) and using (23), we have

$$\tilde{\mathbf{E}}_L = \tilde{\mathbf{E}}_{L-1} + \left(\mathbf{T}_L^H\mathbf{H}_L^H\mathbf{H}_L\mathbf{T}_L + \mathbf{R}_L^{-1}\right)^{-1} \quad (68)$$

where $\tilde{\mathbf{E}}_{L-1} = \mathbf{W}^{\frac{H}{2}}(\mathbf{I}_{N_0} + \mathbf{A}_{L-1}^H\mathbf{C}_{L-1}^{-1}\mathbf{A}_{L-1})^{-1}\mathbf{W}^{\frac{1}{2}}$ is the WMMSE matrix at the $(L-1)$-th hop.

It can be seen from (68) that $\tilde{\mathbf{E}}_L$ can be decomposed recursively. By replacing L with l, we can get $\mathbf{\Psi}_{l-1}$ and $\mathbf{T}_l$ in a similar way as (58)–(68). It can be shown that the optimal $\mathbf{F}_l$ is given by $\mathbf{F}_l = \mathbf{T}_l\mathbf{W}^{\frac{H}{2}}\mathbf{A}_{l-1}^H\mathbf{D}_{l-1}^{-1}$, $l = 2,\ldots,L-1$, and $\tilde{\mathbf{E}}_l$ is given by

$$\tilde{\mathbf{E}}_l = \tilde{\mathbf{E}}_{l-1} + \left(\mathbf{T}_l^H\mathbf{H}_l^H\mathbf{H}_l\mathbf{T}_l + \mathbf{R}_l^{-1}\right)^{-1}, l = 2,\ldots,L-1 \quad (69)$$

$$\tilde{\mathbf{E}}_1 = \mathbf{W}^{\frac{H}{2}}(\mathbf{I}_{N_0} + \mathbf{F}_1^H\mathbf{H}_1^H\mathbf{H}_1\mathbf{F}_1)^{-1}\mathbf{W}^{\frac{1}{2}}. \quad (70)$$

Combining (68)–(70), we obtain $\tilde{\mathbf{E}}_L = \mathbf{W}^{\frac{H}{2}}(\mathbf{I}_{N_0} + \mathbf{F}_1^H\mathbf{H}_1^H\mathbf{H}_1\mathbf{F}_1)^{-1}\mathbf{W}^{\frac{1}{2}} + \sum_{l=2}^{L}(\mathbf{T}_l^H\mathbf{H}_l^H\mathbf{H}_l\mathbf{T}_l + \mathbf{R}_l^{-1})^{-1}$.

Using (21), the transmission power consumed by each relay node in (8) can be rewritten as

$$tr(\mathbf{F}_l\mathbf{D}_{l-1}\mathbf{F}_l^H) = tr(\mathbf{T}_l\mathbf{R}_l\mathbf{T}_l^H), \quad l = 2,\ldots,L. \quad (71)$$

From (22) and (71), the problem (18)–(20) can be equivalently rewritten as the problem (24)–(26).

Acknowledgment

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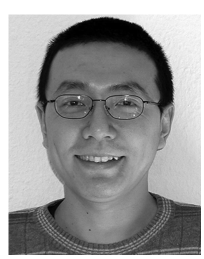
Zhiqiang He (S'01–M'04) received the B.E. degree and Ph.D. degree (with distinction) from Beijing University of Posts and Telecommunications, China, all in signal and information processing, in 1999 and 2004, respectively. Since July 2004, he has been with the School of Information and Communication Engineering, Beijing University of Posts and Telecommunications, where he is currently an Associate Professor and the director of the Center of Information Theory and Technology.

His research interests include signal and information processing in wireless communications, networking architecture and protocol design, and underwater acoustic communications.

Sichuan Guo received the B.E. degree in communication engineering from Beijing University of Posts and Telecommunications, China, in 2012. He is currently working toward the M.S. degree in the School of Information and Communication Engineering, Beijing University of Posts and Telecommunications. He majors in signal and information processing. His expertise and general interests span the areas of communications, signal processing, and channel estimation theory.

Yuanbiao Ou received the B.E. degree from Beijing University of Posts and Telecommunications, China, in 2012. He is currently working toward the M.Eng. degree in the School of Information and Communication Engineering, Beijing University of Posts and Telecommunications. His research interests are in signal and information processing and wireless communications, with emphasis on channel estimation and equalization techniques.



Yue Rong (S'03–M'06–SM'11) received the Ph.D. degree (*summa cum laude*) in electrical engineering from the Darmstadt University of Technology, Darmstadt, Germany, in 2005.

He was a Post-Doctoral Researcher with the Department of Electrical Engineering, University of California, Riverside, from February 2006 to November 2007. Since December 2007, he has been with the Department of Electrical and Computer Engineering, Curtin University, Bentley, Australia, where he is currently an Associate Professor. His research interests include signal processing for communications, wireless communications, underwater acoustic communications, applications of linear algebra and optimization methods, and statistical and array signal processing.

Dr. Rong was a recipient of the Best Paper Award at the 2011 International Conference on Wireless Communications and Signal Processing, the Best Paper Award at the 2010 Asia-Pacific Conference on Communications, and the Young Researcher of the Year Award of the Faculty of Science and Engineering at Curtin University in 2010. He is an Associate Editor of IEEE TRANSACTIONS ON SIGNAL PROCESSING, an Editor of IEEE WIRELESS COMMUNICATIONS LETTERS, a Guest Editor of the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS special issue on Theories and Methods for Advanced Wireless Relays, and was a TPC Member for the IEEE ICC, WCSP, IWCMC, and ChinaCom.