Joint Source and Relay Optimization for Two-Way Linear Non-Regenerative MIMO Relay Communications

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***Abstract*—In this paper, we investigate the challenging problem of joint source and relay optimization for two-way linear non-regenerative multiple-input multiple-output (MIMO) relay communication systems. We derive the optimal structure of the source and relay precoding matrices when linear minimal mean-squared error (MMSE) receivers are used at both destinations in the relay system. We show that for a broad class of frequently used objective functions for MIMO communications such as the MMSE, the maximal mutual information (MMI), and the minimax MSE, the optimal relay and source matrices have a general beamforming structure. This result includes existing works as special cases. Based on this optimal structure, a new iterative algorithm is developed to jointly optimize the relay and source matrices. We also propose a novel suboptimal relay precoding matrix design which significantly reduces the computational complexity of the optimal design with only a marginal performance degradation. Interestingly, we show that this suboptimal relay matrix is indeed optimal for some special cases. The performance of the proposed algorithms are demonstrated by numerical simulations. It is shown that the novel minimax MSE-based two-way relay system has a better bit-error-rate (BER) performance compared with existing two-way relay systems using the MMSE and the MMI criteria.**

***Index Terms*—Beamforming, linear non-regenerative relay, MIMO relay, two-way relay.**

I. Introduction

RELAY communication is well known for being a cost-effective approach to improve the energy-efficiency of wireless communications systems [1]. When nodes in the relay network are equipped with multiple antennas, we have a multiple-input multiple-output (MIMO) relay system [2]–[7]. For a one-way MIMO relay system (i.e., one source node sends information to one destination node), a unified framework has been established in [3] to jointly optimize the source and relay precoding matrices for a broad class of frequently used objective functions in MIMO communications. In [4], the source and relay matrices were jointly optimized for a one-way MIMO relay system where the direct source-destination link is non-negligible. The optimal power allocation in a multiuser MIMO relay system has been investigated in [5]. Recently, one-way and two-way relay systems with multiple parallel MIMO relay nodes have been investigated in [6].

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In a two-way relay communication system, two source nodes exchange their information through an assisting relay node. By resorting to the idea of analog network coding [8], the information exchange can be completed in two time slots. In the first time slot, both source nodes concurrently transmit signals to the relay node. In the second time slot, the relay node precodes the received signals and broadcasts the precoded signals to both source nodes. Since each node knows its own transmitted signals, the self-interference can be easily cancelled. Then the message from the other node can be decoded.

Distributed space-time coding has been designed in [9] for two-way relay communication with multiple single-antenna relay nodes. For a two-way (and in general N-way) relay system with a multi-antenna relay node and single-antenna source nodes, the relay beamforming issue has been investigated in [10] and [11]. Beamforming algorithms have been developed in [12] and [13] for a two-way relay network with multiple single-antenna relay nodes. For two-way MIMO relay systems, the optimal relay and source matrices have been developed in [14] and [15] to maximize the two-way sum mutual information (SMI). Minimal mean-squared error (MMSE) based two-way MIMO relay systems were proposed in [16] and [17]. An algebraic norm-maximization relaying algorithm has been developed in [18]. Two-way relay communication in a multiuser scenario was recently studied in [19] and [20]. An overview on the topics of two-way MIMO relay communication can be found in [21].

In this paper, we investigate the joint source and relay precoding matrices optimization for a two-way MIMO relay communication system where both source nodes and the relay node are equipped with multiple antennas. Compared with existing works such as [10]–[21], the contributions of this paper can be summarized as follows. Firstly, we develop a unified framework for optimizing two-way linear non-regenerative MIMO relay systems. This framework includes a broad class of frequently used Schur-concave and Schur-convex objective functions for MIMO relay system design, while existing works only focus on one specific criterion. Thus, this paper includes existing works as special cases. Moreover, for the first time, we develop a two-way MIMO relay system using the criterion which minimizes the maximum of the MSE (minimax MSE) of signal waveform estimation among all data streams[1].

[1]To the best of our knowledge, the minimax MSE design criterion has been considered only for one-way MIMO relay communications. For two-way relay systems with single-antenna source nodes, the minimax symbol-error rate principle has been recently applied for relay selection in [22]. However, minimax MSE-based two-way MIMO relay system design has not been considered by existing works.

Fig. 1. Block diagram of a two-way non-regenerative MIMO relay communication system.

Secondly, for a broad class of frequently used Schur-concave and Schur-convex objective functions, we show that the optimal relay and source matrices have a general beamforming structure. This interesting outcome includes the results in existing works such as [10] and [14] as special cases. Based on this optimal structure, an iterative algorithm is developed to jointly optimize the relay and source matrices. Note that a rank-constrained optimization problem with a Schur-convex/Schur-concave objective function and multiple trace/logdeterminant constraints is addressed in [23]. However, the joint source, relay, and receiver matrices optimization problem in this paper is more challenging than the problem in [23], since the former problem involves multiple matrix variables, while the latter one only deals with a single matrix variable.

Thirdly, we propose a new suboptimal relay precoding matrix design which significantly reduces the computational complexity of the optimal design with only a marginal performance degradation. Interestingly, we show that this suboptimal relay matrix is indeed optimal for some special cases.

To study the performance of the joint source and relay matrices design algorithms, numerical simulations are carried out using the following three objective functions: (1) The minimal sum MSE (MSMSE) of the signal waveform estimation; (2) The maximal two-way SMI (MSMI); (3) The minimax MSE of the signal waveform estimation. It is shown that the proposed iterative algorithm converges in only a few iterations, which is important for practical two-way relay systems. We also show that the minimax MSE-based relay algorithm has a better bit-error-rate (BER) performance compared with the commonly used MSMSE and MSMI criteria. In this paper, for notational convenience, we consider a narrow band single-carrier system. However, our results can be straightforwardly generalized to each subcarrier of a broadband multi-carrier two-way MIMO relay systems. For multi-carrier two-way relay systems with single antenna nodes, the optimal spectrum sharing and power allocation issue has been studied in [24].

The rest of this paper is organized as follows. In Section II, we introduce the model of a two-way linear non-regenerative MIMO relay communication system. In Section III, we show that the optimal linear receivers are the MMSE receivers. The joint source and relay matrices design algorithms are developed in Section IV. In Section V, we show some numerical examples. Conclusions are drawn in Section VI.

II. System Model

We consider a three-node MIMO communication system where nodes 1 and 2 exchange information with the aid of one relay node as shown in Fig. 1. We assume that both nodes 1 and 2 are equipped with N antennas, the relay node has M antennas[2]. The information exchange between nodes 1 and 2 is completed in two time slots. In the first time slot, nodes 1 and 2 concurrently transmit, and the signal vector from node i is $\mathbf{x}_i = \mathbf{B}_i\mathbf{s}_i, i = 1, 2$, where $\mathbf{s}_i$ is the $N_b \times 1$ source signal vector, and $\mathbf{B}_i$ is the $N \times N_b$ source precoding matrix at node i. Here N_b is the number of information-carrying symbols. Note that when $N_b = 1$, we have a beamforming vector $\mathbf{b}_i$ at node i. The signal vector $\mathbf{y}_r$ received at the relay node can be written as

$$\mathbf{y}_r = \mathbf{H}_{r,1}\mathbf{B}_1\mathbf{s}_1 + \mathbf{H}_{r,2}\mathbf{B}_2\mathbf{s}_2 + \mathbf{v}_r \tag{1}$$

where $\mathbf{H}_{r,i}$, $i = 1, 2$, is the $M \times N$ channel matrix between the relay node and node i, and $\mathbf{v}_r$ is the $M \times 1$ noise vector at the relay node.

In the second time slot, the relay node linearly precodes $\mathbf{y}_r$ with an $M \times M$ matrix $\mathbf{F}$ and broadcasts the precoded signal vector $\mathbf{x}_r = \mathbf{F}\mathbf{y}_r$ to nodes 1 and 2. Using (1), the received signal vector at node i can be written as

$$\begin{aligned}\tilde{\mathbf{y}}_i = \mathbf{H}_{i,r}\mathbf{F}\mathbf{H}_{r,\bar{i}}\mathbf{B}_{\bar{i}}\mathbf{s}_{\bar{i}} + \mathbf{H}_{i,r}\mathbf{F}\mathbf{H}_{r,i}\mathbf{B}_i\mathbf{s}_i \\ +\mathbf{H}_{i,r}\mathbf{F}\mathbf{v}_r + \mathbf{v}_i, \quad i = 1, 2\end{aligned} \tag{2}$$

where $\mathbf{H}_{i,r}$, $i = 1, 2$, is the $N \times M$ channel matrix between node i and the relay node, and $\mathbf{v}_i$, $i = 1, 2$, is the $N \times 1$ noise vector at node i. Here $\bar{i} = 2$ for $i = 1$, and $\bar{i} = 1$ for $i = 2$.

We assume that the source signal vectors satisfy $\mathrm{E}[\mathbf{s}_i\mathbf{s}_i^H] = \mathbf{I}_{N_b}$, $i = 1, 2$, and all noises are independent and identically distributed (i.i.d.) additive white Gaussian noise (AWGN) with zero mean and unit variance. Here $\mathrm{E}[\cdot]$ stands for the statistical expectation, $\mathbf{I}_n$ is an $n \times n$ identity matrix, and $(\cdot)^H$ denotes matrix (vector) Hermitian transpose. In this paper, we assume that all MIMO channels are quasi-static, that is, they remain constant (deterministic) over one time frame, but can change to another value in the next time frame. Such quasi-static channel model has been widely used in one-way and two-way MIMO relay communications [3]–[7], [10]–[14]. We also assume that all three nodes know the channel state information (CSI) of $\mathbf{H}_{r,i}$ and $\mathbf{H}_{i,r}$, $i = 1, 2$, for example, through channel training and estimation [25]. The relay node performs the optimization of $\mathbf{F}$, $\mathbf{B}_1, \mathbf{B}_2$, and then transmits them to nodes 1 and 2. Together with the knowledge of $\mathbf{H}_{r,i}$ and $\mathbf{H}_{i,r}$, $i = 1, 2$, node i can then compute equivalent channel matrices such as $\mathbf{H}_{i,r}\mathbf{F}\mathbf{H}_{r,i}\mathbf{B}_i$ which are necessary for signal reception and self-interference cancellation.

Since node i knows its own transmitted signal vector $\mathbf{s}_i$ and $\mathbf{H}_{i,r}\mathbf{F}\mathbf{H}_{r,i}\mathbf{B}_i$, the self-interference component in (2) can be easily cancelled. The effective received signal vectors are given by

$$\begin{aligned}\mathbf{y}_i &= \mathbf{H}_{i,r}\mathbf{F}\mathbf{H}_{r,\bar{i}}\mathbf{B}_{\bar{i}}\mathbf{s}_{\bar{i}} + \mathbf{H}_{i,r}\mathbf{F}\mathbf{v}_r + \mathbf{v}_i \\ &\triangleq \tilde{\mathbf{H}}_i\mathbf{s}_{\bar{i}} + \tilde{\mathbf{v}}_i, \quad i = 1, 2\end{aligned} \tag{3}$$

where $\tilde{\mathbf{H}}_i \triangleq \mathbf{H}_{i,r}\mathbf{F}\mathbf{H}_{r,\bar{i}}\mathbf{B}_{\bar{i}}$, $i = 1, 2$, is the equivalent MIMO channel seen at node i, and $\tilde{\mathbf{v}}_i \triangleq \mathbf{H}_{i,r}\mathbf{F}\mathbf{v}_r + \mathbf{v}_i$ is the equivalent noise vector at node i.

Due to their lower computational complexity, linear receivers are used at nodes 1 and 2 to retrieve the transmitted signals sent from the other node, and we have $N_b \le \min(M, N)$. The estimated signal waveform vector is given by $\hat{\mathbf{s}}_1 = \mathbf{W}_2^H\mathbf{y}_2$ and $\hat{\mathbf{s}}_2 = \mathbf{W}_1^H\mathbf{y}_1$, where $\mathbf{W}_1$ and $\mathbf{W}_2$ are $N \times N_b$ weight matrices. Note that a linear receiver is suboptimal when $N_b > 1$.

[2]We assume that $M, N \ge 2$. The case of $N = 1$ has been addressed in [10].

However, it reduces the complexity drastically compared with a joint ML detection.

III. Optimal Linear Receiver Matrices

From (3), the MSE matrices of the signal waveform estimation $\mathbf{E}_i(\mathbf{W}_i, \mathbf{F}, \mathbf{B}_{\bar{i}}) \triangleq \mathrm{E}[(\hat{\mathbf{s}}_{\bar{i}} - \mathbf{s}_{\bar{i}})(\hat{\mathbf{s}}_{\bar{i}} - \mathbf{s}_{\bar{i}})^H]$, $i = 1, 2$, is a function of $\mathbf{W}_i$, $\mathbf{F}$, and $\mathbf{B}_{\bar{i}}$, and can be written as

$$\mathbf{E}_i(\mathbf{W}_i, \mathbf{F}, \mathbf{B}_{\bar{i}}) = \left(\mathbf{W}_i^H \tilde{\mathbf{H}}_i - \mathbf{I}_{N_b}\right)\left(\mathbf{W}_i^H \tilde{\mathbf{H}}_i - \mathbf{I}_{N_b}\right)^H + \mathbf{W}_i^H \mathbf{C}_{\tilde{v}_i} \mathbf{W}_i, \quad i = 1, 2 \quad (4)$$

where $\mathbf{C}_{\tilde{v}_i} \triangleq \mathrm{E}[\tilde{\mathbf{v}}_i \tilde{\mathbf{v}}_i^H] = \mathbf{H}_{i,r}\mathbf{F}\mathbf{F}^H\mathbf{H}_{i,r}^H + \mathbf{I}_N$, $i = 1, 2$, is the equivalent noise covariance matrix at node i. Using (4), the MSE of the signal waveform estimation of the jth data stream at node i is given by the (j, j)-th element of $\mathbf{E}_i(\mathbf{W}_i, \mathbf{F}, \mathbf{B}_{\bar{i}})$ as

$$\mathrm{mse}_{i,j} = \mathrm{E}[|\hat{s}_{\bar{i},j} - s_{\bar{i},j}|^2] = [\mathbf{E}_i(\mathbf{W}_i, \mathbf{F}, \mathbf{B}_{\bar{i}})]_{j,j} \quad i = 1, 2, \quad j = 1, \ldots, N_b. \quad (5)$$

In this paper, we aim at minimizing

$$q(\{\mathrm{mse}_{1,j}\}) + q(\{\mathrm{mse}_{2,j}\}) \quad (6)$$

where $\{\mathrm{mse}_{i,j}\} \triangleq [\mathrm{mse}_{i,1}, \ldots, \mathrm{mse}_{i,N_b}]^T$ contains the MSEs of all N_b data streams at node i, $(\cdot)^T$ denotes matrix (vector) transpose, and the multivariate function $q(\cdot)$ is increasing in each one of its arguments while having the rest fixed. Using (5), the objective function (6) can be equivalently rewritten as

$$\min_{\mathbf{W}_1, \mathbf{W}_2, \mathbf{B}_1, \mathbf{B}_2, \mathbf{F}} q(\mathbf{d}[\mathbf{E}_1(\mathbf{W}_1, \mathbf{F}, \mathbf{B}_2)]) + q(\mathbf{d}[\mathbf{E}_2(\mathbf{W}_2, \mathbf{F}, \mathbf{B}_1)]) \quad (7)$$

where for a matrix $\mathbf{A}$, $\mathbf{d}[\mathbf{A}]$ is a column vector containing all main diagonal elements of $\mathbf{A}$.

We consider the following transmission power constraint at the relay node

$$\mathrm{tr}\left(\mathbf{F}\left(\sum_{i=1}^{2} \mathbf{H}_{r,i}\mathbf{B}_i\mathbf{B}_i^H\mathbf{H}_{r,i}^H + \mathbf{I}_M\right)\mathbf{F}^H\right) \le P_r \quad (8)$$

where $\mathrm{tr}(\cdot)$ denotes matrix trace, and P_r is the power available at the relay node. The transmission power constraint at two source nodes can be written as

$$\mathrm{tr}\left(\mathbf{B}_i\mathbf{B}_i^H\right) \le P_i, \quad i = 1, 2 \quad (9)$$

where P_i is the available power at the ith source node. It can be seen that $\mathbf{W}_1$ and $\mathbf{W}_2$ are not in constraints (8) and (9).

The joint optimization over $\mathbf{F}$, $\mathbf{B}_i$, and $\mathbf{W}_i$, $i = 1, 2$, in (7) subjecting to (8) and (9) can be performed over two stages. It is well-known in optimization theory ([32], Sec. 4.1.3) that for *any* function $f(\cdot)$ (not necessarily convex), $\min_{\mathbf{x},\mathbf{y}} f(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{x}} \min_{\mathbf{y}} f(\mathbf{x}, \mathbf{y})$. In other words, we can always minimize a function by first minimizing over some of the variables, and then minimizing over the remaining ones. Based on this fact, we have

$$\begin{aligned}&\min_{\mathbf{W}_1, \mathbf{W}_2, \mathbf{B}_1, \mathbf{B}_2, \mathbf{F} \in (8),(9)} (q(\mathbf{d}[\mathbf{E}_1(\mathbf{W}_1, \mathbf{F}, \mathbf{B}_2)]) + q(\mathbf{d}[\mathbf{E}_2(\mathbf{W}_2, \mathbf{F}, \mathbf{B}_1)])) \\ &= \min_{\mathbf{B}_1, \mathbf{B}_2, \mathbf{F} \in (8),(9)} \left(\min_{\mathbf{W}_1} q(\mathbf{d}[\mathbf{E}_1(\mathbf{W}_1, \mathbf{F}, \mathbf{B}_2)]) + \min_{\mathbf{W}_2} q(\mathbf{d}[\mathbf{E}_2(\mathbf{W}_2, \mathbf{F}, \mathbf{B}_1)])\right). \end{aligned} \quad (10)$$

Interestingly, for two unconstrained inner minimization problems in (10), the optimal $\mathbf{W}_i$ does not depend on the specific form of $q(\cdot)$. The reason is twofold. First, for fixed $\mathbf{F}$ and $\mathbf{B}_{\bar{i}}$, the minimization of $\mathrm{mse}_{i,j}$ with respect to $\mathbf{w}_{i,j}$ (the jth column of $\mathbf{W}_i$) does not incur any penalty on the other substreams. Second, $q(\cdot)$ is increasing in each one of its arguments. Therefore, we can simultaneously minimize all MSEs. For any fixed $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{F}$, two inner minimization problems in (10) are convex quadratic problems and the optimal $\mathbf{W}_i$ is the Wiener filter [26] given by

$$\mathbf{W}_i^{\circ} = \left(\tilde{\mathbf{H}}_i\tilde{\mathbf{H}}_i^H + \mathbf{C}_{\tilde{v}_i}\right)^{-1}\tilde{\mathbf{H}}_i, \quad i = 1, 2 \quad (11)$$

where $(\cdot)^{-1}$ stands for matrix inversion. In fact, $\mathrm{mse}_{i,j}$ (5) can also be written as

$$\begin{aligned}\mathrm{mse}_{i,j} &= \mathrm{E}\left[\left|\mathbf{w}_{i,j}^H\mathbf{y}_i - s_{\bar{i},j}\right|^2\right] \\ &= \mathbf{w}_{i,j}^H\mathrm{E}\left[\mathbf{y}_i\mathbf{y}_i^H\right]\mathbf{w}_{i,j} - \mathbf{w}_{i,j}^H\mathrm{E}[\mathbf{y}_i s_{\bar{i},j}] \\ &\quad - (\mathrm{E}[\mathbf{y}_i s_{\bar{i},j}])^H\mathbf{w}_{i,j} + 1.\end{aligned}$$

The vector $\mathbf{w}_{i,j}$ minimizing $\mathrm{mse}_{i,j}$ above is given by $\mathbf{w}_{i,j}^{\circ} = (\mathrm{E}[\mathbf{y}_i\mathbf{y}_i^H])^{-1}\mathrm{E}[\mathbf{y}_i s_{\bar{i},j}]$. Since $\mathrm{E}[\mathbf{y}_i\mathbf{y}_i^H] = \tilde{\mathbf{H}}_i\tilde{\mathbf{H}}_i^H + \mathbf{C}_{\tilde{v}_i}$ and $\mathrm{E}[\mathbf{y}_i s_{\bar{i},j}] = \tilde{\mathbf{h}}_{i,j}$, we have $\mathbf{w}_{i,j}^{\circ} = (\tilde{\mathbf{H}}_i\tilde{\mathbf{H}}_i^H + \mathbf{C}_{\tilde{v}_i})^{-1}\tilde{\mathbf{h}}_{i,j}$. Here $\tilde{\mathbf{h}}_{i,j}$ is the jth column of $\tilde{\mathbf{H}}_i$. By putting $\mathbf{w}_{i,j}^{\circ}$, $j = 1, \ldots, N_b$, together into one matrix whose jth column is $\mathbf{w}_{i,j}^{\circ}$, we obtain $\mathbf{W}_i^{\circ}$ in (11). This proves that $\mathbf{W}_i^{\circ}$ (11) indeed simultaneously minimizes all diagonal elements of $\mathbf{E}_i$.

After the optimal $\mathbf{W}_i$ is obtained, the outer minimization problem in (10) can be written as

$$\min_{\mathbf{B}_1, \mathbf{B}_2, \mathbf{F}} \quad q\left(\mathbf{d}\left[\mathbf{E}_1^{\circ}(\mathbf{F}, \mathbf{B}_2)\right]\right) + q\left(\mathbf{d}\left[\mathbf{E}_2^{\circ}(\mathbf{F}, \mathbf{B}_1)\right]\right) \quad (12)$$

$$\text{s.t.} \quad \mathrm{tr}\left(\mathbf{F}\left(\sum_{i=1}^{2}\mathbf{H}_{r,i}\mathbf{B}_i\mathbf{B}_i^H\mathbf{H}_{r,i}^H + \mathbf{I}_M\right)\mathbf{F}^H\right) \le P_r \quad (13)$$

$$\mathrm{tr}\left(\mathbf{B}_i\mathbf{B}_i^H\right) \le P_i, \quad i = 1, 2 \quad (14)$$

where $\mathbf{E}_i^{\circ}(\mathbf{F}, \mathbf{B}_{\bar{i}}) \triangleq \mathbf{E}_i(\mathbf{W}_i^{\circ}, \mathbf{F}, \mathbf{B}_{\bar{i}})$, $i = 1, 2$, is the MSE matrix using $\mathbf{W}_i^{\circ}$. By substituting (11) back into (4), we have

$$\begin{aligned}\mathbf{E}_i^{\circ}(\mathbf{F}, \mathbf{B}_{\bar{i}}) &= \mathbf{I}_{N_b} - \tilde{\mathbf{H}}_i^H\left(\tilde{\mathbf{H}}_i\tilde{\mathbf{H}}_i^H + \mathbf{C}_{\tilde{v}_i}\right)^{-1}\tilde{\mathbf{H}}_i \\ &= \left[\mathbf{I}_{N_b} + \tilde{\mathbf{H}}_i^H\mathbf{C}_{\tilde{v}_i}^{-1}\tilde{\mathbf{H}}_i\right]^{-1}, \quad i = 1, 2 \quad (15)\end{aligned}$$

where the second equation is obtained by applying the matrix inversion lemma $(\mathbf{A}+\mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{DA}^{-1}\mathbf{B}+\mathbf{C}^{-1})^{-1}\mathbf{DA}^{-1}$. After the optimal $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{F}$ are obtained from solving the problem (12)–(14), we get the optimal receiver matrices $\mathbf{W}_1^{\text{o}}$ and $\mathbf{W}_2^{\text{o}}$ from (11).

The two-stage optimization approach has the following advantages. Firstly, the receiver matrices in (11) are always optimal for any fixed $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{F}$. Secondly, using the optimal receiver matrices (11) reduces the dimension of variables in the problem of (7)–(9). As a result, the outer minimization problem (12)–(14) needs only to focus on the optimization of $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{F}$. This enables us to identify the optimal structure of $\mathbf{F}$ and $\mathbf{B}_i$ as shown later.

It has been shown in [3] that a broad class of frequently used MIMO relay system design objectives can be written as a function of the main diagonal elements of the MSE matrix $\mathbf{E}_i^{\text{o}}(\mathbf{F},\mathbf{B}_{\bar{i}})$ in (15). To illustrate this interesting link, let us look at the following three examples. First, the sum of the MSE of all data streams can be written as

$$\begin{aligned}\text{MSE}_i &= \text{tr}\left(\left[\mathbf{I}_{N_b} + \tilde{\mathbf{H}}_i^H \mathbf{C}_{\tilde{v}_i}^{-1}\tilde{\mathbf{H}}_i\right]^{-1}\right)\\ &= \sum_{j=1}^{N_b} [\mathbf{E}_i^{\text{o}}(\mathbf{F},\mathbf{B}_{\bar{i}})]_{j,j}. \end{aligned}\tag{16}$$

Here $[\mathbf{E}_i^{\text{o}}(\mathbf{F},\mathbf{B}_{\bar{i}})]_{j,j}$ is the jth main diagonal element of $\mathbf{E}_i^{\text{o}}(\mathbf{F},\mathbf{B}_{\bar{i}})$, and is in fact the MSE of the signal waveform estimation of the jth data stream given by

$$\begin{aligned}[\mathbf{E}_i^{\text{o}}(\mathbf{F},\mathbf{B}_{\bar{i}})]_{j,j} &= \left|\left(\mathbf{w}_{i,j}^{\text{o}}\right)^H \tilde{\mathbf{h}}_{i,j} - 1\right|^2 + \left(\mathbf{w}_{i,j}^{\text{o}}\right)^H \mathbf{C}_{i,j}\mathbf{w}_{i,j}^{\text{o}}\\ &= 1 - \tilde{\mathbf{h}}_{i,j}^H\left(\tilde{\mathbf{h}}_{i,j}\tilde{\mathbf{h}}_{i,j}^H + \mathbf{C}_{i,j}\right)^{-1}\tilde{\mathbf{h}}_{i,j}\\ &= \left[1 + \tilde{\mathbf{h}}_{i,j}^H \mathbf{C}_{i,j}^{-1}\tilde{\mathbf{h}}_{i,j}\right]^{-1}, \quad j=1,\ldots,N_b\end{aligned}$$

where $\mathbf{C}_{i,j} = \mathbf{C}_{\tilde{v}_i} + \tilde{\mathbf{H}}_i\tilde{\mathbf{H}}_i^H - \tilde{\mathbf{h}}_{i,j}\tilde{\mathbf{h}}_{i,j}^H$ is the interference-plus-noise covariance matrix for the jth data stream. Here the matrix inversion lemma is used to obtain the third equation.

Second, the negative MI (NMI) between source and destination is

$$\text{NMI}_i = -\log_2\left|\mathbf{I}_{N_b} + \tilde{\mathbf{H}}_i^H \mathbf{C}_{\tilde{v}_i}^{-1}\tilde{\mathbf{H}}_i\right| \tag{17}$$

where $|\cdot|$ denotes matrix determinant. Since (17) is invariant to any unitary rotation of $\mathbf{B}_i$, we can choose $\mathbf{B}_i$ such that $\tilde{\mathbf{H}}_i^H \mathbf{C}_{\tilde{v}_i}^{-1}\tilde{\mathbf{H}}_i$ is diagonal. Thus we have

$$\text{NMI}_i = \sum_{i=1}^{N_b} \log_2 [\mathbf{E}_i^{\text{o}}(\mathbf{F},\mathbf{B}_{\bar{i}})]_{j,j}. \tag{18}$$

Finally, the maximum of the MSE (MaxMSE) of the signal waveform estimation among all data streams can be written as

$$\text{MaxMSE}_i = \max_j \left([\mathbf{E}_i^{\text{o}}(\mathbf{F},\mathbf{B}_{\bar{i}})]_{j,j}\right). \tag{19}$$

From (16), (18), and (19) we see that all three functions are strongly linked to the main diagonal elements of $\mathbf{E}_i^{\text{o}}(\mathbf{F},\mathbf{B}_{\bar{i}})$. Moreover, it has been shown in [3] that (16) and (18) are Schur-concave functions [28] with respect to $\mathbf{d}[\mathbf{E}_i^{\text{o}}(\mathbf{F},\mathbf{B}_{\bar{i}})]$, and (19) is a Schur-convex function [28] with respect to $\mathbf{d}[\mathbf{E}_i^{\text{o}}(\mathbf{F},\mathbf{B}_{\bar{i}})]$. In this paper, we consider $q(\cdot)$ which includes commonly used Schur-concave and Schur-convex objective functions. As examples, we have

$$\begin{aligned}&q\left(\mathbf{d}\left[\mathbf{E}_1^{\text{o}}(\mathbf{F},\mathbf{B}_2)\right]\right) + q\left(\mathbf{d}\left[\mathbf{E}_2^{\text{o}}(\mathbf{F},\mathbf{B}_1)\right]\right)\\ &\quad = \text{tr}\left(\mathbf{E}_1^{\text{o}}(\mathbf{F},\mathbf{B}_2)\right) + \text{tr}\left(\mathbf{E}_2^{\text{o}}(\mathbf{F},\mathbf{B}_1)\right)\\ &q\left(\mathbf{d}\left[\mathbf{E}_1^{\text{o}}(\mathbf{F},\mathbf{B}_2)\right]\right) + q\left(\mathbf{d}\left[\mathbf{E}_2^{\text{o}}(\mathbf{F},\mathbf{B}_1)\right]\right)\\ &\quad = \log_2\left|\mathbf{E}_1^{\text{o}}(\mathbf{F},\mathbf{B}_2)\right| + \log_2\left|\mathbf{E}_2^{\text{o}}(\mathbf{F},\mathbf{B}_1)\right|\\ &q\left(\mathbf{d}\left[\mathbf{E}_1^{\text{o}}(\mathbf{F},\mathbf{B}_2)\right]\right) + q\left(\mathbf{d}\left[\mathbf{E}_2^{\text{o}}(\mathbf{F},\mathbf{B}_1)\right]\right)\\ &\quad = \max_j\left([\mathbf{E}_1^{\text{o}}(\mathbf{F},\mathbf{B}_2)]_{j,j}\right) + \max_j\left([\mathbf{E}_2^{\text{o}}(\mathbf{F},\mathbf{B}_1)]_{j,j}\right)\end{aligned}$$

when (16), (18), and (19) are chosen as the objective functions, respectively. Due to the limit of space, we only listed three objective functions as examples. Please refer to [29] for a list of commonly used Schur-concave/Schur-convex objective functions in MIMO systems. For an objective function which has not been studied in existing works for two-way MIMO relay system, the optimal source and relay precoding matrices can be obtained by using the unified framework developed in the next section, as long as the objective function is Schur-concave/Schur-convex. The Schur-convexity/Schur-concavity of $q(\cdot)$ will be exploited in Section IV-B for a simplified relay matrix design. It will also be used in Section IV-C to obtain the optimal source precoding matrices.

IV. Joint Source and Relay Optimization

In this section, we develop algorithms to solve the joint source and relay optimization problem (12)–(14) where $q(\cdot)$ is Schur-concave/Schur-convex. Since the problem (12)–(14) is nonconvex, a globally optimal solution of $\mathbf{F}$, $\mathbf{B}_1$, $\mathbf{B}_2$ is difficult to obtain with a reasonable computational complexity (non-exhaustive searching). We develop an iterative algorithm to optimize (12). First we show the optimal structure of $\mathbf{F}$.

A. Optimal Structure of Relay Precoding Matrix

For any feasible $\mathbf{B}_1$ and $\mathbf{B}_2$ satisfying (14), the relay precoding matrix optimization problem is given by

$$\min_{\mathbf{F}} \quad q\left(\mathbf{d}\left[\mathbf{E}_1^{\text{o}}(\mathbf{F})\right]\right) + q\left(\mathbf{d}\left[\mathbf{E}_2^{\text{o}}(\mathbf{F})\right]\right) \tag{20}$$

$$\text{s.t.} \quad \text{tr}\left(\mathbf{F}\left(\sum_{i=1}^{2}\mathbf{H}_{r,i}\mathbf{B}_i\mathbf{B}_i^H\mathbf{H}_{r,i}^H + \mathbf{I}_M\right)\mathbf{F}^H\right) \leq P_r. \tag{21}$$

First we consider the scenario where $M \geq 2N$. The case of $N_b \leq M < 2N$ will be discussed later. Let us introduce the following singular value decompositions (SVDs)

$$\mathbf{H}_1 \triangleq [\mathbf{H}_{r,2}\mathbf{B}_2, \mathbf{H}_{r,1}\mathbf{B}_1] = \mathbf{U}_1\mathbf{\Sigma}_1\mathbf{V}_1^H \tag{22}$$

$$\mathbf{H}_2 \triangleq \left[\mathbf{H}_{1,r}^T, \mathbf{H}_{2,r}^T\right]^T = \mathbf{U}_2\mathbf{\Sigma}_2\mathbf{V}_2^H \tag{23}$$

where the dimensions of $\mathbf{U}_1$, $\mathbf{\Sigma}_1$, $\mathbf{V}_1$ are $M\times 2N_b$, $2N_b\times 2N_b$, $2N_b\times 2N_b$, respectively, and the dimensions of $\mathbf{U}_2$, $\mathbf{\Sigma}_2$, $\mathbf{V}_2$ are $2N\times 2N$, $2N\times 2N$, $M\times 2N$, respectively. Note that $\mathbf{H}_1$ is in fact the equivalent first-hop multiaccess MIMO channel from both source nodes to the relay node, while $\mathbf{H}_2$ is actually the equivalent second-hop broadcast MIMO channel from the relay node to both nodes 1 and 2. The following theorem establishes the optimal structure of $\mathbf{F}$ when $M \geq 2N$.

Theorem 1: Using the SVDs (22) and (23) and a $2N \times 2N_b$ matrix $\mathbf{A}$, the optimal $\mathbf{F}$ as the solution to the problem (20)–(21) is given by

$$\mathbf{F} = \mathbf{V}_2 \mathbf{A} \mathbf{U}_1^H. \tag{24}$$

Proof: See Appendix A. □

Theorem 1 shows that the optimal relay precoding matrix can be viewed as a general form of beamforming. The relay first performs receive beamforming using the Hermitian transpose of the left singular matrix of the effective source-relay channel $\mathbf{H}_1$ (22). Then the relay conducts a linear precoding operation using $\mathbf{A}$. Finally, a transmit beamforming is performed by the relay using the right singular matrix of the relay-destination channel $\mathbf{H}_2$ (23).

In the case of $N = 1$ (or $N_b = 1$) and the MSMI objective, a similar result has been presented in [10], where $\mathbf{A}$ is a 2×2 matrix. Interestingly, (24) extends the result in [10] from the case of single antenna to the scenario of multiple antennas at both source nodes. Moreover, (24) generalizes the results in [10] and [14] from the MSMI objective to a broad class of frequently used objective functions for MIMO relay systems. Note that there are two differences between Theorem 1 and the result in [3]. First, using the notation in this paper, $\mathbf{A}$ is a diagonal matrix in [3]. While in Theorem 1, $\mathbf{A}$ is not necessarily diagonal. In fact, $\mathbf{A}$ is a non-square matrix if $N_b \neq N$. Second, $\mathbf{U}_1$ in [3] depends only on the channel matrix. While in Theorem 1, $\mathbf{U}_1$ is a function of both source precoding matrices $\mathbf{B}_1$ and $\mathbf{B}_2$. These differences make optimizing the source and relay matrices in a two-way MIMO relay channel much more challenging than for a one-way MIMO relay system.

By substituting (24) back into (20)–(21) and introducing $\mathbf{U}_2 = [\mathbf{U}_{2,1}^T, \mathbf{U}_{2,2}^T]^T$, $\mathbf{V}_1^H = [\mathbf{V}_{1,1}^H, \mathbf{V}_{1,2}^H]$, the optimization problem (20)–(21) becomes

$$\min_{\mathbf{A}} \quad \sum_{i=1}^{2} q\left(\mathbf{d}\left[\left(\mathbf{I}_{N_b} + \mathbf{V}_{1,i}\boldsymbol{\Sigma}_1\mathbf{A}^H\boldsymbol{\Sigma}_2\mathbf{U}_{2,i}^H\left(\mathbf{U}_{2,i}\boldsymbol{\Sigma}_2\mathbf{A}\mathbf{A}^H \times\boldsymbol{\Sigma}_2\mathbf{U}_{2,i}^H + \mathbf{I}_N\right)^{-1}\mathbf{U}_{2,i}\boldsymbol{\Sigma}_2\mathbf{A}\boldsymbol{\Sigma}_1\mathbf{V}_{1,i}^H\right)^{-1}\right]\right) \tag{25}$$

$$\text{s.t.} \quad \operatorname{tr}\left(\mathbf{A}\left(\boldsymbol{\Sigma}_1^2 + \mathbf{I}_{2N_b}\right)\mathbf{A}^H\right) \leq P_r \tag{26}$$

where the dimensions of $\mathbf{U}_{2,1}$ and $\mathbf{U}_{2,2}$ are $N \times 2N$, and the dimension of $\mathbf{V}_{1,1}$ and $\mathbf{V}_{1,2}$ are $N_b \times 2N_b$. For systems with $M \geq 2N$, since the dimension of $\mathbf{A}$ is smaller than $\mathbf{F}$, solving the problem (25)–(26) has a smaller computational complexity than solving the problem (20)–(21). For relay systems with $N_b \leq M < 2N$, we directly solve the problem (20)–(21) using the projected gradient algorithm similar to that listed in Table I shown later on to obtain (at least) a locally optimal solution of $\mathbf{F}$.

In general, the problem (25)–(26) is nonconvex and a globally optimal solution is difficult to obtain with a reasonable computational complexity (non-exhaustive searching). We can resort to numerical methods, such as the projected gradient algorithm [30] to find (at least) a locally optimal solution of (25)–(26). The advantage of the projected gradient algorithm is that it only requires information on the first-order derivative (gradient) of the objective function. While other approaches for nonlinear programming, such as the sequential quadratic programming (SQP) and the Newton method, require also the knowledge on the second-order derivative. Since the problem (25)–(26) has matrix variable, the complexity of computing the second-order derivative of (25) with respect to $\mathbf{A}$ is much higher than that of the first-order derivative.

TABLE I
PROCEDURE OF APPLYING THE PROJECTED GRADIENT ALGORITHM TO SOLVE THE PROBLEM (25)–(26)

1) Initialize the algorithm at a feasible $\mathbf{A}^{(0)}$; Set $n = 0$.
2) Compute the gradient of (25) $\nabla f(\mathbf{A}^{(n)})$;
   Project $\tilde{\mathbf{A}}^{(n)} = \mathbf{A}^{(n)} - t_n \nabla f(\mathbf{A}^{(n)})$ to the set of $\operatorname{tr}\big(\bar{\mathbf{A}}^{(n)}(\boldsymbol{\Sigma}_1^2 + \mathbf{I}_{2N_b})(\bar{\mathbf{A}}^{(n)})^H\big) \leq P_r$ to obtain $\bar{\mathbf{A}}^{(n)}$;
   Update $\mathbf{A}$ with $\mathbf{A}^{(n+1)} = \mathbf{A}^{(n)} + \gamma_n\big(\bar{\mathbf{A}}^{(n)} - \mathbf{A}^{(n)}\big)$.
3) If $\max \operatorname{abs}\big(\mathbf{A}^{(n+1)} - \mathbf{A}^{(n)}\big) \leq \varepsilon$, then end.
   Otherwise, let $n := n + 1$ and go to step 2).

In the projected gradient algorithm, we need to compute the gradient of the objective function (25). As an example, if $f(\mathbf{A}) = \operatorname{tr}(\mathbf{E}_1^{\text{o}}(\mathbf{A})) + \operatorname{tr}(\mathbf{E}_2^{\text{o}}(\mathbf{A}))$ is chosen as the objective function, then its gradient $\nabla f(\mathbf{A})$ with respect to $\mathbf{A}$ can be calculated by using results on derivatives of matrices in [31] as

$$\nabla f(\mathbf{A}) = -2\sum_{i=1}^{2}\Big[\mathbf{Z}_i\boldsymbol{\Sigma}_1\mathbf{V}_{1,i}^H\left[\mathbf{I}_{N_b} + \mathbf{V}_{1,i}\boldsymbol{\Sigma}_1(\mathbf{I}_{2N_b} - \mathbf{Z}_i) \times\boldsymbol{\Sigma}_1\mathbf{V}_{1,i}^H\right]^{-2}\mathbf{V}_{1,i}\boldsymbol{\Sigma}_1\mathbf{Z}_i\mathbf{A}^H\boldsymbol{\Sigma}_2\mathbf{U}_{2,i}^H\mathbf{U}_{2,i}\boldsymbol{\Sigma}_2\Big]^H$$

where $\mathbf{Z}_i \triangleq (\mathbf{I}_{2N_b} + \mathbf{A}^H\boldsymbol{\Sigma}_2\mathbf{U}_{2,i}^H\mathbf{U}_{2,i}\boldsymbol{\Sigma}_2\mathbf{A})^{-1}$, $i = 1, 2$. Then we obtain $\tilde{\mathbf{A}} \triangleq \mathbf{A} - t\nabla f(\mathbf{A})$ by moving $\mathbf{A}$ one step towards the negative gradient direction of $f(\mathbf{A})$, where $t > 0$ is the step size. Since $\tilde{\mathbf{A}}$ might not satisfy the constraint (22), we need to project it onto the set given by (26). The projected matrix $\bar{\mathbf{A}}$ is obtained by minimizing the Frobenius norm of $\bar{\mathbf{A}} - \tilde{\mathbf{A}}$ (according to [30]) subjecting to (26), which can be formulated as the following optimization problem

$$\begin{aligned} \min_{\bar{\mathbf{A}}} \quad & \operatorname{tr}\left((\bar{\mathbf{A}} - \tilde{\mathbf{A}})(\bar{\mathbf{A}} - \tilde{\mathbf{A}})^H\right) \\ \text{s.t.} \quad & \operatorname{tr}\left(\bar{\mathbf{A}}\left(\boldsymbol{\Sigma}_1^2 + \mathbf{I}_{2N_b}\right)\bar{\mathbf{A}}^H\right) \leq P_r. \end{aligned} \tag{27}$$

Obviously, if $\operatorname{tr}(\tilde{\mathbf{A}}(\boldsymbol{\Sigma}_1^2 + \mathbf{I}_{2N_b})\tilde{\mathbf{A}}^H) \leq P_r$, then $\bar{\mathbf{A}} = \tilde{\mathbf{A}}$. Otherwise, the solution to the problem (27) can be obtained by using the Lagrange multiplier method and given by

$$\bar{\mathbf{A}} = \tilde{\mathbf{A}}\left[(\lambda + 1)\mathbf{I}_{2N_b} + \lambda\boldsymbol{\Sigma}_1^2\right]^{-1}$$

where $\lambda > 0$ is the solution to the nonlinear equation of

$$\operatorname{tr}\left(\tilde{\mathbf{A}}\left[(\lambda + 1)\mathbf{I}_{2N_b} + \lambda\boldsymbol{\Sigma}_1^2\right]^{-1}\left(\boldsymbol{\Sigma}_1^2 + \mathbf{I}_{2N_b}\right) \times\left[(\lambda + 1)\mathbf{I}_{2N_b} + \lambda\boldsymbol{\Sigma}_1^2\right]^{-1}\tilde{\mathbf{A}}^H\right) = P_r. \tag{28}$$

Since (28) is a monotonically decreasing function of λ, the unique solution of (28) can be efficiently obtained by the bisection method [30].

The procedure of the projected gradient algorithm is listed in Table I, where t_n and γ_n are the step size parameters at the nth iteration, $\max\mathrm{abs}(\cdot)$ denotes the maximum among the absolute value of all elements in a matrix, and ε is a positive constant close to 0. The step size parameters t_n and γ_n are chosen by the Armijo rule [30], i.e., $t_n = t$ is a constant through all iterations, while at the nth iteration, γ_n is set to be β^{m_n}. Here m_n is the minimal nonnegative integer that satisfies the inequality of $f(\mathbf{A}^{(n+1)}) - f(\mathbf{A}^{(n)}) \leq \alpha\beta^{m_n}\mathrm{tr}((\nabla f(\mathbf{A}^{(n)}))^H(\bar{\mathbf{A}}^{(n)} - \mathbf{A}^{(n)}))$, α and β are constants. According to [30], usually α is chosen close to 0, for example $\alpha \in [10^{-5}, 10^{-1}]$, and a proper choice of β is normally from 0.1 to 0.5.

B. Simplified Relay Matrix Design

In this subsection, we focus on relay systems with $M \geq 2N$ and develop a relay precoding matrix design algorithm which is suboptimal for general cases, but has a much lower computational complexity than directly solving the problem (25)–(26). We will show that this suboptimal relay matrix is indeed optimal for some special cases. Let us introduce

$$\boldsymbol{\Sigma}_2\mathbf{A} = \begin{bmatrix}\mathbf{U}_{2,1}^H, & \mathbf{U}_{2,2}^H\end{bmatrix}\begin{bmatrix}\mathbf{C}_1\\ \mathbf{C}_2\end{bmatrix} = \mathbf{U}_{2,1}^H\mathbf{C}_1 + \mathbf{U}_{2,2}^H\mathbf{C}_2 \quad (29)$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are $N \times 2N_b$ matrices. Since $[\mathbf{U}_{2,1}^H, \quad \mathbf{U}_{2,2}^H] = \mathbf{U}_2^H$ is a unitary matrix, for any $\mathbf{A}$, we have $[\mathbf{C}_1^T, \quad \mathbf{C}_2^T]^T = \mathbf{U}_2\boldsymbol{\Sigma}_2\mathbf{A}$. Thus, instead of optimizing $\mathbf{A}$, one can equivalently optimize $\mathbf{C}_1$ and $\mathbf{C}_2$. Substituting (29) back into (25), we obtain that $\mathbf{U}_{2,i}\boldsymbol{\Sigma}_2\mathbf{A} = \mathbf{C}_i$, $i = 1, 2$, and thus

$$\mathbf{E}_i^{\circ}(\mathbf{C}_i) = \left[\mathbf{I}_{N_b} + \mathbf{V}_{1,i}\boldsymbol{\Sigma}_1\mathbf{C}_i^H\left(\mathbf{C}_i\mathbf{C}_i^H + \mathbf{I}_N\right)^{-1}\mathbf{C}_i\boldsymbol{\Sigma}_1\mathbf{V}_{1,i}^H\right]^{-1} \quad i = 1, 2. \quad (30)$$

Interestingly, it can be seen from (30) that $\mathbf{E}_i^{\circ}$ is only a function of $\mathbf{C}_i$, $i = 1, 2$. In other words, the optimization variables are decoupled for $\mathbf{E}_1^{\circ}(\mathbf{C}_1)$ and $\mathbf{E}_2^{\circ}(\mathbf{C}_2)$.

Let us introduce the SVDs of

$$\boldsymbol{\Sigma}_1\mathbf{V}_{1,i}^H = \mathbf{P}_i\boldsymbol{\Pi}_i\mathbf{R}_i^H, \quad i = 1, 2 \quad (31)$$

where $\boldsymbol{\Pi}_i$, $i = 1, 2$, are $N_b \times N_b$ diagonal matrices, $\mathbf{R}_i$, $i = 1, 2$, are $N_b \times N_b$ unitary matrices, and $\mathbf{P}_i$, $i = 1, 2$, are $2N_b \times N_b$ semi-unitary matrices with $\mathbf{P}_i^H\mathbf{P}_i = \mathbf{I}_{N_b}$. Substituting (31) back into (30), we have

$$\mathbf{E}_i^{\circ}(\mathbf{C}_i) = \Big[\mathbf{I}_{N_b} + \mathbf{R}_i\boldsymbol{\Pi}_i\mathbf{P}_i^H\mathbf{C}_i^H\left(\mathbf{C}_i\mathbf{C}_i^H + \mathbf{I}_N\right)^{-1} \times \mathbf{C}_i\mathbf{P}_i\boldsymbol{\Pi}_i\mathbf{R}_i^H\Big]^{-1}, \quad i = 1, 2. \quad (32)$$

Let us introduce the SVD of $\mathbf{C}_i$ as

$$\mathbf{C}_i = \mathbf{U}_{c_i}\boldsymbol{\Delta}_i\mathbf{V}_{c_i}^H, \quad i = 1, 2 \quad (33)$$

where $\boldsymbol{\Delta}_i$ is an $N_b \times N_b$ diagonal singularvalue matrix. It can be easily seen that $\mathbf{U}_{c_i}$ is irrelevant to (32) and can be any $N \times N_b$ semi-unitary matrix ($\mathbf{U}_{c_i}^H\mathbf{U}_{c_i} = \mathbf{I}_{N_b}$). However, $\mathbf{U}_{c_i}$ affects the power constraint (26) as explained later. Now we show that the optimal $\mathbf{V}_{c_i}$ for the objective function (25) is given by

$$\mathbf{V}_{c_i} = \mathbf{P}_i, \quad i = 1, 2. \quad (34)$$

In fact, for any $\mathbf{B}_1$ and $\mathbf{B}_2$ in (22), one can always have $\bar{\mathbf{B}}_1 = \mathbf{B}_1\mathbf{R}_2$ and $\bar{\mathbf{B}}_2 = \mathbf{B}_2\mathbf{R}_1$ such that the objective function (25) with $\bar{\mathbf{B}}_1$ and $\bar{\mathbf{B}}_2$ (i.e., $\mathbf{E}_i^{\circ}(\mathbf{C}_i)$ becomes $\mathbf{R}_i^H\mathbf{E}_i^{\circ}(\mathbf{C}_i)\mathbf{R}_i$, $i = 1, 2$) is equal to $\sum_{i=1}^2 q(\mathbf{d}[\mathbf{E}_i^{\circ}(\mathbf{C}_i)])$, where

$$\begin{aligned}\mathbf{E}_i^{\circ}(\mathbf{C}_i) &= \left[\mathbf{I}_{N_b} + \boldsymbol{\Pi}_i\mathbf{P}_i^H\mathbf{C}_i^H\left(\mathbf{C}_i\mathbf{C}_i^H + \mathbf{I}_N\right)^{-1}\mathbf{C}_i\mathbf{P}_i\boldsymbol{\Pi}_i\right]^{-1}\\ &= \Big[\mathbf{I}_{N_b} + \boldsymbol{\Pi}_i\mathbf{P}_i^H\mathbf{V}_{c_i}\boldsymbol{\Delta}_i\left(\boldsymbol{\Delta}_i^2 + \mathbf{I}_{N_b}\right)^{-1}\\ &\qquad \times \boldsymbol{\Delta}_i\mathbf{V}_{c_i}^H\mathbf{P}_i\boldsymbol{\Pi}_i\Big]^{-1}, \quad i = 1, 2.\end{aligned} \quad (35)$$

For the objective function $q(\cdot)$, which is Schur-concave with respect to $\mathbf{d}[\mathbf{E}_i^{\circ}(\mathbf{C}_i)]$ such as (16) and (18), it can be shown similar to [3] that the optimal $\mathbf{V}_{c_i}$ for (35) are given by (34). In this case, $\mathbf{E}_1^{\circ}(\mathbf{C}_1)$ and $\mathbf{E}_2^{\circ}(\mathbf{C}_2)$ in (35) are diagonalized by $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively as $\mathbf{E}_i^{\circ} = [\mathbf{I}_{N_b} + \boldsymbol{\Pi}_i\boldsymbol{\Delta}_i(\boldsymbol{\Delta}_i^2 + \mathbf{I}_{N_b})^{-1}\boldsymbol{\Delta}_i\boldsymbol{\Pi}_i]^{-1}$, $i = 1, 2$, and the objective function (25) can be written as

$$\sum_{i=1}^{2} q\left(\mathbf{d}\left[\left(\mathbf{I}_{N_b} + \boldsymbol{\Pi}_i\boldsymbol{\Delta}_i\left(\boldsymbol{\Delta}_i^2 + \mathbf{I}_{N_b}\right)^{-1}\boldsymbol{\Delta}_i\boldsymbol{\Pi}_i\right)^{-1}\right]\right). \quad (36)$$

For a Schur-convex $q(\cdot)$ such as (19), it can also be shown from [3] that the optimal $\mathbf{V}_{c_i}$ are given by (34). In this case, for any $\mathbf{B}_1$ and $\mathbf{B}_2$ in (22), the final source precoding matrices are taken as $\breve{\mathbf{B}}_1 = \mathbf{B}_1\mathbf{R}_2\mathbf{U}_{b_2}$ and $\breve{\mathbf{B}}_2 = \mathbf{B}_2\mathbf{R}_1\mathbf{U}_{b_1}$, where $\mathbf{U}_{b_i}$ is an $N_b \times N_b$ unitary matrix making $\mathbf{E}_i^{\circ} = [\mathbf{I}_{N_b} + \mathbf{U}_{b_i}^H\boldsymbol{\Pi}_i\boldsymbol{\Delta}_i(\boldsymbol{\Delta}_i^2 + \mathbf{I}_{N_b})^{-1}\boldsymbol{\Delta}_i\boldsymbol{\Pi}_i\mathbf{U}_{b_i}]^{-1}$ have identical diagonal entries [3]. Consequently, for all Schur-convex $q(\cdot)$, we only need to minimize $\sum_{i=1}^2 \mathrm{tr}([\mathbf{I}_{N_b} + \boldsymbol{\Pi}_i\boldsymbol{\Delta}_i(\boldsymbol{\Delta}_i^2 + \mathbf{I}_{N_b})^{-1}\boldsymbol{\Delta}_i\boldsymbol{\Pi}_i]^{-1})$. In other words, $q(\cdot)$ in (36) is taken as the summation over all its variables for all Schur-convex $q(\cdot)$.

Now we consider the power constraint (26). From (29), $\mathbf{A}$ is given by

$$\mathbf{A} = \boldsymbol{\Sigma}_2^{-1}\mathbf{U}_2^H\begin{bmatrix}\mathbf{C}_1\\ \mathbf{C}_2\end{bmatrix}. \quad (37)$$

Substituting (33) and (34) into (37), which is then substituted back into (26), the transmission power consumed by the relay node can be written as

$$\begin{aligned}&\mathrm{tr}\left(\mathbf{A}\left(\boldsymbol{\Sigma}_1^2 + \mathbf{I}_{2N_b}\right)\mathbf{A}^H\right)\\ &\quad = \mathrm{tr}\left(\boldsymbol{\Sigma}_2^{-1}\mathbf{U}_2^H\,\mathrm{bd}(\mathbf{U}_{c_1}\boldsymbol{\Delta}_1, \mathbf{U}_{c_2}\boldsymbol{\Delta}_2)\boldsymbol{\Phi}\right.\\ &\qquad \left.\times\mathrm{bd}\left(\boldsymbol{\Delta}_1\mathbf{U}_{c_1}^H, \boldsymbol{\Delta}_2\mathbf{U}_{c_2}^H\right)\mathbf{U}_2\boldsymbol{\Sigma}_2^{-1}\right)\end{aligned} \quad (38)$$

where $\mathrm{bd}(\cdot)$ stands for a block diagonal matrix, and $\boldsymbol{\Phi} \triangleq [\mathbf{P}_1, \quad \mathbf{P}_2]^H(\boldsymbol{\Sigma}_1^2 + \mathbf{I}_{2N_b})[\mathbf{P}_1, \quad \mathbf{P}_2]$. From (36) and (38), the relay precoding matrix optimization problem (25)–(26) is converted to the following problem

$$\min_{\boldsymbol{\Delta}_1,\boldsymbol{\Delta}_2,\mathbf{U}_{c_1},\mathbf{U}_{c_2}} \quad \sum_{i=1}^{2} q\left(\mathbf{d}\left[\left(\mathbf{I}_{N_b}+\boldsymbol{\Pi}_i^2\boldsymbol{\Delta}_i^2\left(\boldsymbol{\Delta}_i^2+\mathbf{I}_{N_b}\right)^{-1}\right)^{-1}\right]\right) \tag{39}$$

$$\text{s.t.} \quad \mathrm{tr}\left(\boldsymbol{\Sigma}_2^{-1}\mathbf{U}_2^H\,\mathrm{bd}\left(\mathbf{U}_{c_1}\boldsymbol{\Delta}_1,\mathbf{U}_{c_2}\boldsymbol{\Delta}_2\right)\boldsymbol{\Phi} \times\mathrm{bd}\left(\boldsymbol{\Delta}_1\mathbf{U}_{c_1}^H,\boldsymbol{\Delta}_2\mathbf{U}_{c_2}^H\right)\mathbf{U}_2\boldsymbol{\Sigma}_2^{-1}\right)\le P_r \tag{40}$$

$$\mathbf{U}_{c_i}^H\mathbf{U}_{c_i}=\mathbf{I}_{N_b},\quad i=1,2 \tag{41}$$

$$\delta_{i,n}\ge 0,\quad i=1,2,\quad n=1,\ldots,N_b \tag{42}$$

where $\delta_{i,n}$, $i=1,\ldots,N_b$, denotes the nth main diagonal element of $\boldsymbol{\Delta}_i$, $i=1,2$.

Note that although $\mathbf{V}_{c_i}$ in (34) is optimal for the objective function (25), we can not prove the optimality of (34) for the constraint function (26). This is the reason that this relay precoding matrix design is suboptimal for general cases. However, compared with the problem (25)–(26), the dimension of optimization variables in the problem (39)–(42) has reduced from $8NN_b$ real numbers to $4NN_b+2N_b$ real numbers, which is significant especially when N is large. It will be shown in Section V that the suboptimal design by solving (39)–(42) has only a marginal increase of MSE compared with solving (25)–(26) directly using the projected gradient algorithm. Such performance-complexity tradeoff is very important for practical two-way MIMO relay systems.

The problem (39)–(42) is nonconvex due to the unitary matrix constraints in (41). Before we develop a numerical method to solve this problem, let us have some insights into the structure of this suboptimal relay precoding matrix. Interestingly, we will show that for two special cases, the suboptimal relay matrix is indeed optimal. By substituting (33) and (34) into (37), which is then substituted back into (24), we obtain

$$\mathbf{F}=\mathbf{V}_2\boldsymbol{\Sigma}_2^{-1}\mathbf{U}_2^H\begin{bmatrix}\mathbf{U}_{c_1}\boldsymbol{\Delta}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{U}_{c_2}\boldsymbol{\Delta}_2\end{bmatrix}\begin{bmatrix}\mathbf{P}_1^H\\ \mathbf{P}_2^H\end{bmatrix}\mathbf{U}_1^H. \tag{43}$$

We can also show from (31) that

$$\begin{bmatrix}\mathbf{P}_1^H\\ \mathbf{P}_2^H\end{bmatrix}\mathbf{U}_1^H=\begin{bmatrix}\boldsymbol{\Pi}_1^{-1}\mathbf{R}_1^H & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Pi}_2^{-1}\mathbf{R}_2^H\end{bmatrix}\begin{bmatrix}\mathbf{V}_{1,1}\\ \mathbf{V}_{1,2}\end{bmatrix}\boldsymbol{\Sigma}_1\mathbf{U}_1^H. \tag{44}$$

Finally, by substituting (44) back into (43), we can equivalently rewrite the relay precoding matrix as

$$\mathbf{F}=\mathbf{V}_2\boldsymbol{\Sigma}_2^{-1}\mathbf{U}_2^H\begin{bmatrix}\mathbf{U}_{c_1}\boldsymbol{\Delta}_1\boldsymbol{\Pi}_1^{-1}\mathbf{R}_1^H & \mathbf{0}\\ \mathbf{0} & \mathbf{U}_{c_2}\boldsymbol{\Delta}_2\boldsymbol{\Pi}_2^{-1}\mathbf{R}_2^H\end{bmatrix} \times\begin{bmatrix}\mathbf{V}_{1,1}\\ \mathbf{V}_{1,2}\end{bmatrix}\times\boldsymbol{\Sigma}_1\mathbf{U}_1^H\triangleq\mathbf{F}_3\mathbf{F}_2\mathbf{F}_1 \tag{45}$$

where

$$\mathbf{F}_3=\mathbf{V}_2\boldsymbol{\Sigma}_2^{-1}\mathbf{U}_2^H$$
$$\mathbf{F}_2=\begin{bmatrix}\mathbf{U}_{c_1}\boldsymbol{\Delta}_1\boldsymbol{\Pi}_1^{-1}\mathbf{R}_1^H & \mathbf{0}\\ \mathbf{0} & \mathbf{U}_{c_2}\boldsymbol{\Delta}_2\boldsymbol{\Pi}_2^{-1}\mathbf{R}_2^H\end{bmatrix}$$
$$\mathbf{F}_1=\begin{bmatrix}\mathbf{V}_{1,1}\\ \mathbf{V}_{1,2}\end{bmatrix}\boldsymbol{\Sigma}_1\mathbf{U}_1^H. \tag{46}$$

Interestingly, it can be seen from (45) and (46) that the relay precoding matrix is composed of three linear filters. First, we know from (22) that $\mathbf{F}_1=\mathbf{H}_1^H$, and hence $\mathbf{F}_1$ is a matched-filter (MF) for the equivalent first-hop multiaccess MIMO channel $\mathbf{H}_1$. Then the signals are linearly filtered by $\mathbf{F}_2$. Finally, we can see from (23) that $\mathbf{F}_3=\mathbf{H}_2^\dagger$, where $(\cdot)^\dagger$ denotes matrix pseudo inverse. Thus, $\mathbf{F}_3$ performs zero-forcing (ZF) of the equivalent second-hop broadcast MIMO channel $\mathbf{H}_2$.

Theorem 2: The structure of $\mathbf{F}$ given by (45) is optimal for the two cases of

$$\text{(a) } \mathcal{R}(\mathbf{H}_{r,1}\mathbf{B}_1)\perp\mathcal{R}(\mathbf{H}_{r,2}\mathbf{B}_2),\mathcal{R}\left(\mathbf{H}_{1,r}^H\right)\perp\mathcal{R}\left(\mathbf{H}_{2,r}^H\right)$$
$$\text{(b) } \mathcal{R}(\mathbf{H}_{r,1}\mathbf{B}_1)\parallel\mathcal{R}(\mathbf{H}_{r,2}\mathbf{B}_2),\mathcal{R}\left(\mathbf{H}_{1,r}^H\right)\perp\mathcal{R}\left(\mathbf{H}_{2,r}^H\right)$$

where $\mathcal{R}(\cdot)$ stands for the range of a matrix.

Proof: See Appendix B. □

It has been shown in [10] that for two-way relay systems with $N=1$ (i.e., both source nodes have only one antenna) and reciprocal first and second hop channels[3] (i.e., $\mathbf{h}_{r,1}=\mathbf{h}_{1,r}^T=\mathbf{h}_1$, $\mathbf{h}_{r,2}=\mathbf{h}_{2,r}^T=\mathbf{h}_2$), both $\mathbf{F}=\mathbf{H}_2^\dagger\begin{bmatrix}a_{\mathrm{ZF}} & 0\\ 0 & b_{\mathrm{ZF}}\end{bmatrix}\mathbf{H}_1^\dagger$ and $\mathbf{F}=\mathbf{H}_2^H\begin{bmatrix}a_{\mathrm{MF}} & 0\\ 0 & b_{\mathrm{MF}}\end{bmatrix}\mathbf{H}_1^H$ are optimal when $\mathbf{h}_1\perp\mathbf{h}_2$, and the latter $\mathbf{F}$ is also optimal when $\mathbf{h}_1\parallel\mathbf{h}_2$. Here a_{ZF}, b_{ZF}, a_{MF}, b_{MF} are scalars that remain to be optimized based on the particular objective function. Interestingly, Theorem 2 extends the result in [10] to two-way relay systems with $N\ge 2$ and without any channel reciprocity, and shows that (45) is optimal for the two special cases given above. We would like to mention that although the two cases listed in Theorem 2 seldom appear in practical systems, they provide important theoretical results.

Now we show how to solve the problem (39)–(42) numerically using the projected gradient algorithm. Since $\mathbf{U}_{c_1}$ and $\mathbf{U}_{c_2}$ only appear in the constraint functions, we can optimize $\mathbf{U}_{c_i}$ and $\boldsymbol{\Delta}_i$ in an alternating fashion. In each iteration, we first optimize $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$ by solving a problem consisting of (39), (40), (42) with fixed $\mathbf{U}_{c_1}$ and $\mathbf{U}_{c_2}$. This problem can be equivalently rewritten as

$$\min_{\boldsymbol{\delta}} \quad \sum_{i=1}^{2} q\left(\left\{\frac{\delta_{i,n}^2+1}{\left(\pi_{i,n}^2+1\right)\delta_{i,n}^2+1}\right\}\right) \tag{47}$$

$$\text{s.t.} \quad \boldsymbol{\delta}^T\left[(\mathbf{L}_1\odot\mathbf{L}_2)^H(\mathbf{L}_1\odot\mathbf{L}_2)\right]\boldsymbol{\delta}\le P_r \tag{48}$$

$$\delta_{i,n}\ge 0,\quad i=1,2,\quad n=1,\ldots,N_b \tag{49}$$

where $\boldsymbol{\delta}\triangleq[\delta_{1,1},\ldots,\delta_{1,N_b},\delta_{2,1},\ldots,\delta_{2,N_b}]^T$, $\odot$ denotes Khatri-Rao product, $\pi_{i,n}$, $n=1,\ldots,N_b$, is the nth main diagonal element of $\boldsymbol{\Pi}_i$, $i=1,2$, $\mathbf{L}_1\triangleq[\mathrm{bd}(\mathbf{U}_{c_1}^H,\mathbf{U}_{c_2}^H)\,\mathbf{U}_2\boldsymbol{\Sigma}_2^{-1}]^T$, $\mathbf{L}_2\triangleq(\boldsymbol{\Sigma}_1^2+\mathbf{I}_{2N_b})^{\frac{1}{2}}[\mathbf{P}_1\quad\mathbf{P}_2]$, and for a scalar a, $\{a_n\}\triangleq[a_1,\ldots,a_{N_b}]^T$. The subproblem (47)–(49) can be solved by the projected gradient algorithm. The gradient of (47) with respect to scalars $\delta_{i,n}$ is easy to obtain. The projection onto the feasible set specified by the quadratic constraint (48) can be performed in a way similar to (27).

[3]For the consistency of notations, here we use vector notations for channels due to $N=1$.

TABLE II
PROCEDURE OF APPLYING THE ALTERNATING PROJECTED GRADIENT ALGORITHM TO SOLVE THE PROBLEM (39)–(42)

1) Initialize the algorithm at a feasible $\mathbf{U}_{c1}^{(0)}$, $\mathbf{U}_{c2}^{(0)}$, and $\boldsymbol{\delta}^{(0)}$; Set $n=0$.
2) With given $\mathbf{U}_{c1}^{(n)}$ and $\mathbf{U}_{c2}^{(n)}$, obtain $\boldsymbol{\delta}^{(n+1)}$ by solving the subproblem (47)-(49) using the projected gradient algorithm similar to the procedure in Table I;
Obtain $\mathbf{U}_{c1}^{(n+1)}$ and $\mathbf{U}_{c2}^{(n+1)}$ through solving the subproblem (50)-(51) with known $\boldsymbol{\delta}^{(n+1)}$ using the projected gradient algorithm.
3) If $\max \mathrm{abs}\big(\boldsymbol{\delta}^{(n+1)} - \boldsymbol{\delta}^{(n)}\big) \le \varepsilon$, then end.
Otherwise, let $n := n+1$ and go to step 2).

With fixed $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$, we update $\mathbf{U}_{c_1}$ and $\mathbf{U}_{c_2}$ by solving the following problem

$$\min_{\mathbf{U}_{c_1},\mathbf{U}_{c_2}} \quad \mathrm{tr}\left(\mathbf{N}_1^H \mathrm{bd}(\mathbf{U}_{c_1},\mathbf{U}_{c_2})\mathbf{N}_2 \mathrm{bd}\left(\mathbf{U}_{c_1}^H,\mathbf{U}_{c_2}^H\right)\mathbf{N}_1\right) \tag{50}$$

$$\text{s.t.} \quad \mathbf{U}_{c_i}^H\mathbf{U}_{c_i} = \mathbf{I}_{N_b}, \quad i=1,2 \tag{51}$$

where $\mathbf{N}_1 \triangleq \mathbf{U}_2\boldsymbol{\Sigma}_2^{-1}$, $\mathbf{N}_2 \triangleq \mathrm{bd}(\boldsymbol{\Delta}_1,\boldsymbol{\Delta}_2)\,\boldsymbol{\Phi}\,\mathrm{bd}(\boldsymbol{\Delta}_1,\boldsymbol{\Delta}_2)$, and the objective function (50) is obtained by rewriting the left-hand side of (40). The subproblem (50)–(51) can also be solved by the projected gradient algorithm. The gradient of (50) with respect to $\mathbf{U}_{c_i}$, denoted as $\nabla g_i(\mathbf{U}_{c_1},\mathbf{U}_{c_2})$, $i=1,2$, can be calculated using the results on derivatives of matrices in [31]. The projection of an $N\times N_b$ matrix $\tilde{\mathbf{U}}_{c_i} = \mathbf{U}_{c_i} - \gamma\nabla g_i(\mathbf{U}_{c_1},\mathbf{U}_{c_2})$ onto the feasible set of $\bar{\mathbf{U}}_{c_i}$ given by (51) is performed by solving the following problem for $i=1,2$

$$\min_{\bar{\mathbf{U}}_{c_i}} \ \mathrm{tr}\left((\bar{\mathbf{U}}_{c_i}-\tilde{\mathbf{U}}_{c_i})^H(\bar{\mathbf{U}}_{c_i}-\tilde{\mathbf{U}}_{c_i})\right) \quad \text{s.t. } \bar{\mathbf{U}}_{c_i}^H\bar{\mathbf{U}}_{c_i} = \mathbf{I}_{N_b}. \tag{52}$$

Let $\tilde{\mathbf{U}}_{c_i} = \boldsymbol{\Upsilon}_i\boldsymbol{\Gamma}_i\boldsymbol{\Theta}_i^H$ be the SVD of $\tilde{\mathbf{U}}_{c_i}$. It can be easily shown using the Lagrange multiplier method that the solution to the problem (52) is given by $\bar{\mathbf{U}}_{c_i} = \boldsymbol{\Upsilon}_i\boldsymbol{\Theta}_i^H$. We would like to mention that in contrast to (26), the feasible set specified by (51) is nonconvex. Using $\mathbf{U}_{c_i}^{(n+1)} = \mathbf{U}_{c_i}^{(n)} + \gamma_n(\bar{\mathbf{U}}_{c_i}^{(n)} - \mathbf{U}_{c_i}^{(n)})$ as in the third line of step 2) in Table I may lead to an infeasible $\mathbf{U}_{c_i}^{(n+1)}$. Thus, at the nth iteration, $\mathbf{U}_{c_i}^{(n+1)}$ is directly updated as the projection of $\mathbf{U}_{c_i}^{(n)} - \gamma_n\nabla g_i(\mathbf{U}_{c_1}^{(n)},\mathbf{U}_{c_2}^{(n)})$, where the step size parameter γ_n is taken from the Armijo rule. The procedure of solving the problem (39)–(42) using the alternating projected gradient algorithm is summarized in Table II.

C. Optimal Structure of Source Precoding Matrices

In this subsection, we develop optimal source precoding matrices $\mathbf{B}_1$ and $\mathbf{B}_2$. It can be seen from (15) that $\mathbf{B}_2$ is irrelevant to $\mathbf{E}_2^{\circ}$. Thus, for fixed $\mathbf{F}$ and $\mathbf{B}_2$, the problem of optimizing $\mathbf{B}_1$ is given by

$$\min_{\mathbf{B}_1} \quad q\left(\mathbf{d}\left[\left(\mathbf{I}_{N_b} + \mathbf{B}_1^H\boldsymbol{\Psi}_1\mathbf{B}_1\right)^{-1}\right]\right) \tag{53}$$

$$\text{s.t.} \quad \mathrm{tr}\left(\mathbf{B}_1^H\mathbf{B}_1\right) \le P_1 \tag{54}$$

$$\mathrm{tr}\left(\mathbf{B}_1^H\mathbf{H}_{r,1}^H\mathbf{F}^H\mathbf{F}\mathbf{H}_{r,1}\mathbf{B}_1\right) \le \tilde{P}_r \tag{55}$$

where $\boldsymbol{\Psi}_1 \triangleq \mathbf{H}_{r,1}^H\mathbf{F}^H\mathbf{H}_{2,r}^H\mathbf{C}_{\tilde{v}_2}^{-1}\mathbf{H}_{2,r}\mathbf{F}\mathbf{H}_{r,1}$ and $\tilde{P}_r \triangleq P_r - \mathrm{tr}(\mathbf{F}(\mathbf{H}_{r,2}\mathbf{B}_2\mathbf{B}_2^H\mathbf{H}_{r,2}^H + \mathbf{I}_M)\mathbf{F}^H)$. When $q(\mathbf{d}[\mathbf{E}_i^{\circ}]) = \log_2|\mathbf{E}_i^{\circ}|$, the problem (53)–(55) has been solved in [5]. For the case of $q(\mathbf{d}[\mathbf{E}_i^{\circ}]) = \mathrm{tr}(\mathbf{E}_i^{\circ})$, the problem (53)–(55) is solved in [4]. Using the Lagrange multiplier method, both [4] and [5] reveal that the optimal $\mathbf{B}_1$ has the structure of $\mathbf{B}_1 = \mathbf{M}_1^{-H}\tilde{\mathbf{Q}}_1\mathbf{D}_1$, where $\mathbf{M}_1\mathbf{M}_1^H = \mu_1\mathbf{I}_N + \mu_2\mathbf{H}_{r,1}^H\mathbf{F}^H\mathbf{F}\mathbf{H}_{r,1}$, $\mathbf{D}_1$ is an $N_b\times N_b$ diagonal matrix,

$$\mathbf{M}_1^{-1}\boldsymbol{\Psi}_1\mathbf{M}_1^{-H} = \mathbf{Q}_1\boldsymbol{\Lambda}_1\mathbf{Q}_1^H \tag{56}$$

is the eigenvalue decomposition (EVD) of $\mathbf{M}_1^{-1}\boldsymbol{\Psi}_1\mathbf{M}_1^{-H}$, and $\tilde{\mathbf{Q}}_1$ contains N_b columns of $\mathbf{Q}_1$ associated with the largest N_b eigenvalues. Here $\mu_1 \ge 0$, $\mu_2 \ge 0$ are the Lagrange multipliers, $\mathbf{Q}_1$ is an $N\times N$ eigenvector matrix, and $\boldsymbol{\Lambda}_1$ is an $N\times N$ diagonal eigenvalue matrix.

In fact, if $\mathbf{B}_1$ satisfies (54) and (55), it must also satisfy the constraint of $\mathrm{tr}(\mathbf{B}_1^H\mathbf{M}_1\mathbf{M}_1^H\mathbf{B}_1) \le \mu_1 P_1 + \mu_2\tilde{P}_r$. Introducing $\tilde{\mathbf{B}}_1 \triangleq \mathbf{M}_1^H\mathbf{B}_1$, a relaxed problem of the original problem (53)–(55) is given by

$$\min_{\tilde{\mathbf{B}}_1} \quad q\left(\mathbf{d}\left[\left(\mathbf{I}_{N_b} + \tilde{\mathbf{B}}_1^H\mathbf{M}_1^{-1}\boldsymbol{\Psi}_1\mathbf{M}_1^{-H}\tilde{\mathbf{B}}_1\right)^{-1}\right]\right) \tag{57}$$

$$\text{s.t.} \quad \mathrm{tr}\left(\tilde{\mathbf{B}}_1^H\tilde{\mathbf{B}}_1\right) \le \mu_1 P_1 + \mu_2\tilde{P}_r. \tag{58}$$

It has been shown in [33] that for any Schur-concave objective function $q(\cdot)$, the solution to the problem (57)–(58) is given by $\tilde{\mathbf{B}}_1 = \tilde{\mathbf{Q}}_1\mathbf{D}_1$. While for any Schur-convex $q(\cdot)$ [28], the optimal $\tilde{\mathbf{B}}_1$ is $\tilde{\mathbf{B}}_1 = \tilde{\mathbf{Q}}_1\mathbf{D}_1\mathbf{U}_{b_2}$, where $\mathbf{U}_{b_2}$ is an $N_b\times N_b$ unitary matrix such that $[\mathbf{I}_{N_b} + \tilde{\mathbf{B}}_1^H\mathbf{M}_1^{-1}\boldsymbol{\Psi}_1\mathbf{M}_1^{-H}\tilde{\mathbf{B}}_1]^{-1}$ has identical main diagonal elements [33]. Therefore, for Schur-concave $q(\cdot)$, the optimal $\mathbf{B}_1$ can be written as

$$\mathbf{B}_1 = \mathbf{M}_1^{-H}\tilde{\mathbf{Q}}_1\mathbf{D}_1. \tag{59}$$

While for Schur-convex $q(\cdot)$, we obtain the optimal $\mathbf{B}_1$ as

$$\mathbf{B}_1 = \mathbf{M}^{-H}\tilde{\mathbf{Q}}_1\mathbf{D}_1\mathbf{U}_{b_2}. \tag{60}$$

It can be seen from (59) that if (17) is taken as the objective function, the optimal $\tilde{\mathbf{B}}_1$ diagonalizes $\tilde{\mathbf{B}}_1^H\mathbf{M}_1^{-1}\boldsymbol{\Psi}_1\mathbf{M}_1^{-H}\tilde{\mathbf{B}}_1$. In other words, the optimal $\mathbf{B}_1$ diagonalizes $\tilde{\mathbf{H}}_2^H\mathbf{C}_{\tilde{v}_2}^{-1}\tilde{\mathbf{H}}_2$ in (17). We would like to mention that since (58) is a necessary constraint that should be satisfied by all feasible $\mathbf{B}_1$ and $\mathbf{B}_2$, (59) or (60) provides a necessary structure of the optimal source precoding matrices. Interestingly, both negative MI and MSE are Schur-concave functions [33]. Thus, the problems discussed in [4] and [5] are special cases of the problem (53)–(55), and the results obtained in [4] and [5] are special instances of (59).

It can be seen from (59) and (60) that $\mathbf{B}_1$ has a beamforming structure, where the directions of beams are determined by $\mathbf{M}_1^{-H}\tilde{\mathbf{Q}}_1$ and $\mathbf{D}_1$ represents the power allocation at each beam. The value of μ_1, μ_2, and $\mathbf{D}_1$ depends on the specific expression of the objective function $q(\cdot)$ and can be obtained via solving the dual optimization problem associated with the original problem (53)–(55) as proposed in [4] and [5]. For fixed $\mathbf{F}$ and $\mathbf{B}_1$, the optimal structure of $\mathbf{B}_2$ can be derived similar to (53)–(60).

Interestingly, for the case of $q(\mathbf{d}[\mathbf{E}_i^{\circ}]) = \mathrm{tr}(\mathbf{E}_i^{\circ})$, $i = 1, 2$, and $N = N_b$, we can also update $\mathbf{B}_1$ and $\mathbf{B}_2$ in parallel. The problem of updating $\mathbf{B}_1$ and $\mathbf{B}_2$ in this case can be written as

$$\min_{\mathbf{B}_1,\mathbf{B}_2} \quad \sum_{i=1}^{2} \mathrm{tr}\left(\left[\mathbf{I}_N + \mathbf{B}_i^H \mathbf{\Psi}_i \mathbf{B}_i\right]^{-1}\right) \tag{61}$$

$$\text{s.t.} \quad \mathrm{tr}\left(\sum_{i=1}^{2} \mathbf{F}\mathbf{H}_{r,i}\mathbf{B}_i\mathbf{B}_i^H\mathbf{H}_{r,i}^H\mathbf{F}^H\right) \leq \breve{P}_r \tag{62}$$

$$\mathrm{tr}\left(\mathbf{B}_i\mathbf{B}_i^H\right) \leq P_i, \quad i = 1, 2 \tag{63}$$

where $\breve{P}_r \triangleq P_r - \mathrm{tr}(\mathbf{F}\mathbf{F}^H)$. Let us introduce $\mathbf{\Omega}_i \triangleq \mathbf{B}_i\mathbf{B}_i^H$, $i = 1, 2$, and positive semi-definite (PSD) matrices $\mathbf{X}_i$ with $\mathbf{X}_i \succeq (\mathbf{I}_N + \mathbf{\Psi}_i^{\frac{1}{2}}\mathbf{\Omega}_i\mathbf{\Psi}_i^{\frac{1}{2}})^{-1}$, $i = 1, 2$, where $\mathbf{A} \succeq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is a PSD matrix. By using the Schur complement [32], the problem (61)–(63) can be equivalently converted to the following problem

$$\min_{\mathbf{X}_1,\mathbf{X}_2,\mathbf{\Omega}_1,\mathbf{\Omega}_2} \quad \mathrm{tr}(\mathbf{X}_1 + \mathbf{X}_2) \tag{64}$$

$$\text{s.t.} \quad \begin{pmatrix} \mathbf{X}_i & \mathbf{I}_N \\ \mathbf{I}_N & \mathbf{I}_N + \mathbf{\Psi}_i^{\frac{1}{2}}\mathbf{\Omega}_i\mathbf{\Psi}_i^{\frac{1}{2}} \end{pmatrix} \succeq 0, \quad i = 1, 2 \tag{65}$$

$$\mathrm{tr}\left(\sum_{i=1}^{2} \mathbf{F}\mathbf{H}_{r,i}\mathbf{\Omega}_i\mathbf{H}_{r,i}^H\mathbf{F}^H\right) \leq \breve{P}_r \tag{66}$$

$$\mathrm{tr}(\mathbf{\Omega}_i) \leq P_i, \quad \mathbf{\Omega}_i \succeq 0, \quad i = 1, 2. \tag{67}$$

The problem (64)–(67) is a convex semi-definite programming (SDP) problem which can be efficiently solved by the interior-point method [32]. Let us introduce the EVD of $\mathbf{\Omega}_i = \mathbf{U}_{\Omega_i}\mathbf{\Lambda}_{\Omega_i}\mathbf{U}_{\Omega_i}^H$, $i = 1, 2$. Then from $\mathbf{\Omega}_i = \mathbf{B}_i\mathbf{B}_i^H$, we have $\mathbf{B}_i = \mathbf{U}_{\Omega_i}\mathbf{\Lambda}_{\Omega_i}^{\frac{1}{2}}$.

Now we present the method to find $\mathbf{U}_{b_i}$ in (60) for Schur-convex $q(\cdot)$. According to [3], for all Schur-convex objective functions, since the MSE matrix $\mathbf{E}_i^{\circ}$ has identical diagonal entries, we only need to minimize $\mathrm{tr}(\mathbf{E}_i^{\circ})$, despite the specific form of the objective function $q(\cdot)$. Therefore, for Schur-convex $q(\cdot)$ such as the MaxMSE in (19), the optimal $\mathbf{B}_1$ and $\mathbf{B}_2$ can be obtained in two steps. Let us take $\mathbf{B}_1$ as an example. First, we solve the problem (53)–(55) using $q(\mathbf{d}[\mathbf{E}_2^{\circ}]) = \mathrm{tr}(\mathbf{E}_2^{\circ})$ to obtain (59). In the second step, we rotate $\mathbf{B}_1$ by a unitary matrix $\mathbf{U}_{b_2}$ such that $\mathbf{E}_2^{\circ}$ have identical diagonal elements. Using (56), the rotated $\mathbf{E}_2^{\circ}$ can be written as

$$\begin{aligned}\mathbf{E}_2^{\circ} &= \left(\mathbf{I}_{N_b} + \mathbf{U}_{b_2}^H\mathbf{D}_1\tilde{\mathbf{Q}}_1^H\mathbf{M}_1^{-1}\mathbf{\Psi}_1\mathbf{M}_1^{-H}\tilde{\mathbf{Q}}_1\mathbf{D}_1\mathbf{U}_{b_2}\right)^{-1} \\ &= \mathbf{U}_{b_2}^H\left(\mathbf{I}_{N_b} + \mathbf{D}_1^2\tilde{\mathbf{\Lambda}}_1\right)^{-1}\mathbf{U}_{b_2}\end{aligned} \tag{68}$$

where $\tilde{\mathbf{\Lambda}}_1$ is an $N_b \times N_b$ diagonal matrix containing the largest N_b eigenvalues of $\mathbf{M}_1^{-1}\mathbf{\Psi}_1\mathbf{M}_1^{-H}$ in (56). To make $\mathbf{E}_2^{\circ}$ have identical diagonal elements, $\mathbf{U}_{b_2}$ can be any $N_b \times N_b$ unitary rotation matrix that satisfies $|[\mathbf{U}_{b_2}]_{j,k}| = |[\mathbf{U}_{b_2}]_{j,l}|, \forall j, k, l$. When N_b is appropriate such as a power of two, the discrete Fourier transform matrix can be chosen for $\mathbf{U}_{b_2}$. While for general case, $\mathbf{U}_{b_2}$ can be computed using the method developed in [34].

TABLE III
PROCEDURE OF SOLVING THE PROBLEM (12)–(14)

1) Initialize the algorithm at a feasible $\mathbf{B}_1^{(0)}$ and $\mathbf{B}_2^{(0)}$; Set $n = 0$.
2) For fixed $\mathbf{B}_1^{(n)}$ and $\mathbf{B}_2^{(n)}$, obtain $\mathbf{F}^{(n+1)}$ by solving the problem (25)-(26) using the steps in Table I, or solving the problem (39)-(42) following the procedure in Table II;
Update $\mathbf{B}_1^{(n+1)}$ and $\mathbf{B}_2^{(n+1)}$ through (59) or (60), or by solving the subproblem (64)-(67) with known $\mathbf{F}^{(n+1)}$.
3) If $\max \mathrm{abs}\left(\mathbf{F}^{(n+1)} - \mathbf{F}^{(n)}\right) \leq \varepsilon$, then end.
Otherwise, let $n := n + 1$ and go to step 2).
4) For Schur-convex $q(\cdot)$, rotate $\mathbf{B}_i$ by $\mathbf{U}_{b_i}$ as in (68).

D. Joint Source and Relay Precoding Matrices Optimization

Now the original joint source and relay optimization problem (12)–(14) can be solved by an iterative algorithm. This algorithm is first initialized at random feasible $\mathbf{B}_1$ and $\mathbf{B}_2$ satisfying (14). At each iteration, we first update $\mathbf{F}$ with fixed $\mathbf{B}_1$ and $\mathbf{B}_2$, and then update $\mathbf{B}_1$ and $\mathbf{B}_2$ with fixed $\mathbf{F}$.

When $M \geq 2N$, we can update $\mathbf{F}$ by exploiting its optimal structure in (24), where $\mathbf{A}$ is obtained by solving the problem (25)–(26) using the procedure in Table I. Alternatively, we can also update $\mathbf{F}$ based on the simplified relay matrix design (45), where $\mathbf{\Delta}_1, \mathbf{\Delta}_2, \mathbf{U}_{c_1}, \mathbf{U}_{c_2}$ are obtained by solving the problem (39)–(42) following the steps in Table II. When $N_b \leq M < 2N$, $\mathbf{F}$ is updated by solving the problem (20)–(21) using the projected gradient algorithm similar to that listed in Table I.

The source precoding matrices are updated as follows. With fixed $\mathbf{F}$ and $\mathbf{B}_2$, we update $\mathbf{B}_1$ as (59) or (60) depending on the Schur-convexity of the objective function $q(\cdot)$. Next, $\mathbf{B}_2$ is updated similar to (59) or (60) with fixed $\mathbf{F}$ and $\mathbf{B}_1$. In the case that $N = N_b$ and SMSE is adopted as the objective function, we can also update $\mathbf{B}_1$ and $\mathbf{B}_2$ in parallel by solving the problem (64)–(67).

Note that the conditional updates of each matrix may either decrease or maintain but cannot increase the objective function (12). Monotonic convergence of $\mathbf{F}$, $\mathbf{B}_1$, and $\mathbf{B}_2$ towards (at least) a locally optimal solution follows directly from this observation. Finally, for Schur-convex $q(\cdot)$, we rotate $\mathbf{B}_i$ by $\mathbf{U}_{b_i}$ as in (68). The procedure of this iterative algorithm is summarized in Table III.

V. Numerical Examples

In this section, we study the performance of the proposed algorithms for two-way MIMO relay systems. All channel matrices have complex Gaussian entries with zero-mean and variances of $1/N$, $1/M$ for $\mathbf{H}_{r,i}$ and $\mathbf{H}_{i,r}$, $i = 1, 2$, respectively[4], and all simulation results are averaged over 1000 independent channel realizations. In the simulations, we set $P_1 = P_2 = P_s = 20$ dB above the noise level and vary the value of P_r. The proposed joint source and relay optimization algorithms are applicable for a broad class of frequently used objective functions, and in the simulations, we consider the following three functions as examples: (1) The SMSE of the signal waveform estimation written as $q(\mathbf{d}[\mathbf{E}_1]) + q(\mathbf{d}[\mathbf{E}_2]) = \mathrm{tr}(\mathbf{E}_1 + \mathbf{E}_2)$; (2) The negative two-way SMI given by $q(\mathbf{d}[\mathbf{E}_1]) + q(\mathbf{d}[\mathbf{E}_2]) = \log_2|\mathbf{E}_1| +$

[4]The variances are set to normalize the effect of number of transmit antennas to the receive signal-to-noise ratio.

$\log_2|\mathbf{E}_2|$, which is also adopted in [10] and [14]; (3) The maximum of the MSE of the signal waveform estimation among all data streams as $q(\mathbf{d}[\mathbf{E}_1]) + q(\mathbf{d}[\mathbf{E}_2]) = \max_j([\mathbf{E}_1]_{j,j}) + \max_j([\mathbf{E}_2]_{j,j})$. We refer to them as the MSMSE objective, the MSMI objective, and the minimax MSE objective, respectively. Note that the first two functions are Schur-concave function, while the last function is Schur-convex. The normalized SMSE and the SMI shown in the simulation results are calculated as $\frac{1}{2N_b}\mathrm{tr}(\mathbf{E}_1 + \mathbf{E}_2)$ and $-(\log_2|\mathbf{E}_1| + \log_2|\mathbf{E}_2|)$, respectively.

In the first example, we check the performance of the proposed relay precoding matrix (45) and the algorithm in Table II by testing it for the case of $N = 1$ and $\mathbf{h}_1 \perp \mathbf{h}_2$. It is proven in Theorem 2 that (45), or equivalently

$$\mathbf{F} = \mathbf{H}_2^\dagger \begin{bmatrix} a_f & 0 \\ 0 & b_f \end{bmatrix} \mathbf{H}_1^H \tag{69}$$

is optimal for this case, and we only need to find the optimal a_f and b_f in (69). By substituting (69) back into (20)–(21) and taking $q(\mathbf{d}[\mathbf{E}_i]) = \mathrm{tr}(\mathbf{E}_i)$, we have the following problem to solve

$$\min_{a_f,b_f} \quad \frac{1+c_1|a_f|^2}{1+c_1d_2|a_f|^2} + \frac{1+c_2|b_f|^2}{1+c_2d_1|b_f|^2} \tag{70}$$

$$\text{s.t.} \quad d_2|a_f|^2 + d_1|b_f|^2 \le P_r \tag{71}$$

where $c_i \triangleq \|\mathbf{h}_{i,r}\|^2$ and $d_i \triangleq 1+P_i\|\mathbf{h}_{r,i}\|^2, i = 1,2$. Here $\|\cdot\|$ denotes the vector Euclidean norm. The problem (70)–(71) has a water-filling solution given by

$$|a_f|^2 = \frac{1}{c_1d_2}\left[\sqrt{\frac{c_1}{\kappa_1}\left(1-\frac{1}{d_2}\right)} - 1\right]^+$$

$$|b_f|^2 = \frac{1}{c_2d_1}\left[\sqrt{\frac{c_2}{\kappa_1}\left(1-\frac{1}{d_1}\right)} - 1\right]^+ \tag{72}$$

where $[x]^+ \triangleq \max(x,0)$, and $\kappa_1 > 0$ is the solution to the nonlinear equation of $\frac{1}{c_1}\left[\sqrt{\frac{c_1}{\kappa_1}(1-\frac{1}{d_2})} - 1\right]^+ + \frac{1}{c_2}\left[\sqrt{\frac{c_2}{\kappa_1}(1-\frac{1}{d_1})} - 1\right]^+ = P_r$, which can be efficiently solved by the bisection method [30].

When $q(\mathbf{d}[\mathbf{E}_i]) = \log_2|\mathbf{E}_i|$ is chosen as the objective function, we need to solve the optimization problem of

$$\min_{a_f,b_f} \quad \log_2\left(\frac{1+c_1|a_f|^2}{1+c_1d_2|a_f|^2}\right) + \log_2\left(\frac{1+c_2|b_f|^2}{1+c_2d_1|b_f|^2}\right) \tag{73}$$

$$\text{s.t.} \quad d_2|a_f|^2 + d_1|b_f|^2 \le P_r. \tag{74}$$

The solution to the problem (73)–(74) is given by

$$|a_f|^2 = \frac{1}{2c_1d_2}[\sqrt{(d_2+1)^2 + 4(c_1(d_2-1)/\kappa_2 - d_2)} - d_2 - 1]^+ \tag{75}$$

$$|b_f|^2 = \frac{1}{2c_2d_1}[\sqrt{(d_1+1)^2 + 4(c_2(d_1-1)/\kappa_2 - d_1)} - d_1 - 1]^+ \tag{76}$$

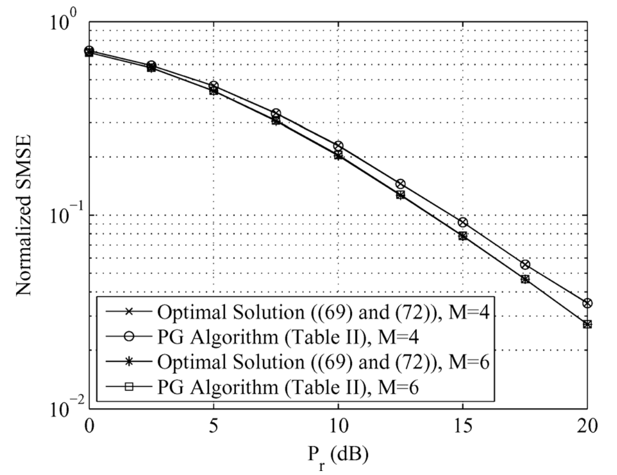

Fig. 2. Example 1: Normalized SMSE versus P_r. $N = 1$.

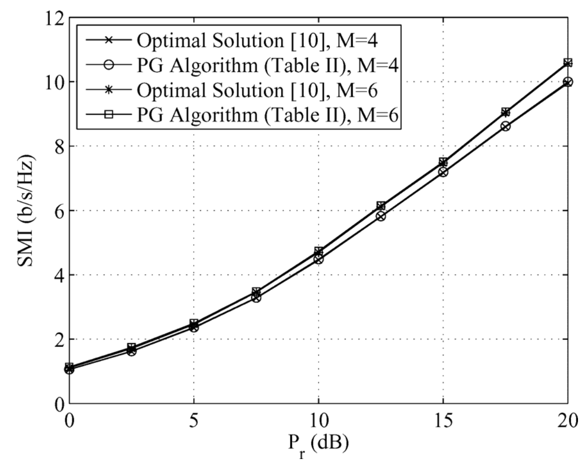

Fig. 3. Example 1: SMI versus P_r. $N = 1$.

where $\kappa_2 > 0$ is the solution to the nonlinear equation of

$$\frac{\left[\sqrt{(d_2+1)^2 + 4\left(\frac{c_1(d_2-1)}{\kappa_2} - d_2\right)} - d_2 - 1\right]^+}{2c_1} + \frac{\left[\sqrt{(d_1+1)^2 + 4\left(\frac{c_2(d_1-1)}{\kappa_2} - d_1\right)} - d_1 - 1\right]^+}{2c_2} = P_r \tag{77}$$

which can be solved by the bisection method.

Fig. 2 shows the normalized SMSE produced by the relay precoding matrix designed by the alternating projected gradient (PG) algorithm in Table II, and that of the optimal solution given by (69) and (72). The $\mathrm{tr}(\mathbf{E}_1) + \mathrm{tr}(\mathbf{E}_2)$ objective is used for both approaches. It can be seen that for both $M = 4$ and $M = 6$, two algorithms have identical SMSE performance. This demonstrates that the algorithm in Table II achieves the global optimum for $N = 1$, and verifies the effectiveness of the projected gradient algorithm. We also observe from Fig. 2 that as expected, the SMSE decreases with increasing M.

Fig. 3 demonstrates the SMI of the system using the relay matrix from the PG algorithm in Table II and that of [10] (which

TABLE IV
NORMALIZED SMSE OF JOINT SOURCE AND RELAY OPTIMIZATION ALGORITHM (TABLE III) AT DIFFERENT ITERATIONS

P_r (dB)	0	2.5	5	7.5	10	12.5	15	17.5	20
Normalized SMSE (It. 1)	0.8233	0.7391	0.6248	0.4936	0.3641	0.2461	0.1723	0.1060	0.0695
Normalized SMSE (It. 2)	0.8232	0.7389	0.6247	0.4935	0.3640	0.2458	0.1712	0.1053	0.0689
Normalized SMSE (It. 5)	0.8228	0.7384	0.6241	0.4934	0.3638	0.2456	0.1698	0.1046	0.0685

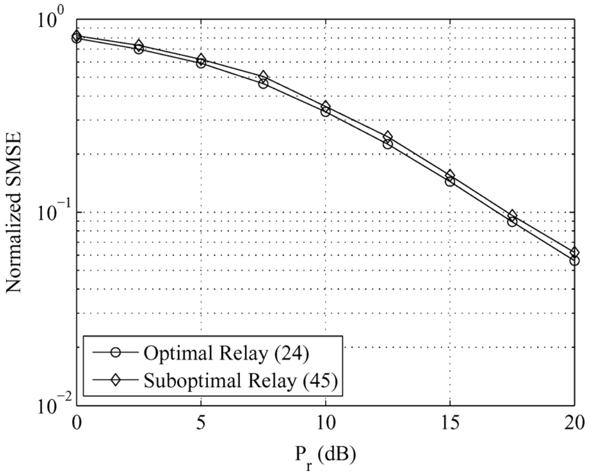

Fig. 4. Example 2: Normalized SMSE versus P_r. $N_b = N = 2$, $M = 8$.

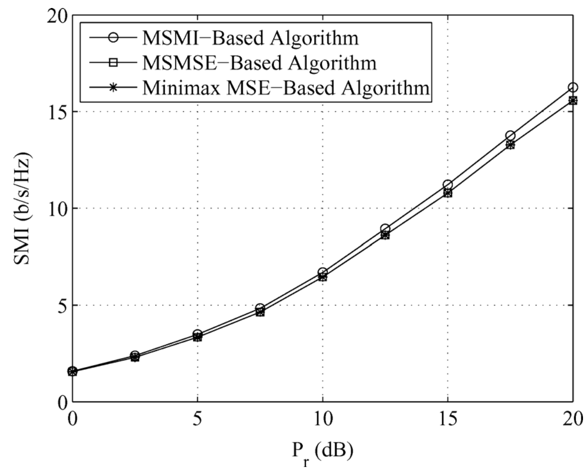

Fig. 5. Example 4: SMI versus P_r. $N_b = N = 2$, $M = 6$.

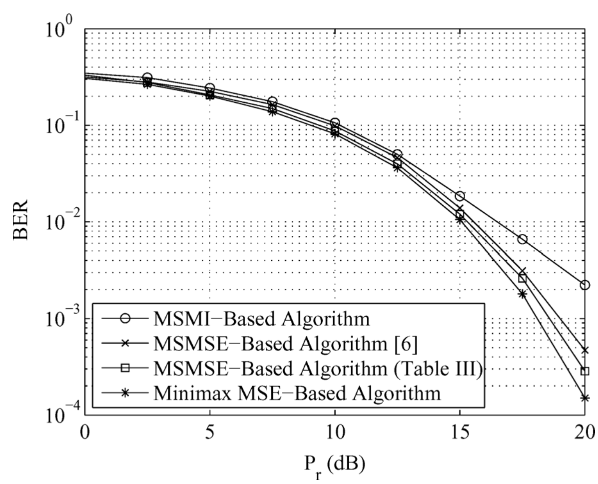

Fig. 6. Example 4: BER versus P_r. $N_b = N = 2$, $M = 6$.

is essentially the solution given by (69), (75), (76)). Both algorithms use $\log_2 |\mathbf{E}_1| + \log_2 |\mathbf{E}_2|$ as the objective function. It is obvious that for both $M = 4$ and $M = 6$, two algorithms have identical SMI performance, indicating that the algorithm in Table II achieves the global optimum in this case.

In the second example, we simulate a two-way MIMO relay system with $N_b = N = 2$ and $M = 8$ using $\mathrm{tr}(\mathbf{E}_1) + \mathrm{tr}(\mathbf{E}_2)$ as the objective function. We compare the normalized SMSE of the optimal relay matrix in (24) using the steps in Table I and the suboptimal relay design in (45) from the procedure listed in Table II. In order to study the "pure" effect of relay matrix design, we set $\mathbf{B}_1 = \mathbf{B}_2 = \sqrt{P_s/2}\,\mathbf{I}_2$ for both algorithms. It can be seen from Fig. 4 that the suboptimal relay precoding matrix yields only a slightly higher MSE than the optimal relay matrix. Since the suboptimal relay matrix design has a substantially reduced computational complexity (20 real-valued optimization variables) than the optimal design (32 real-valued optimization variables), it is very useful in practical relay systems. In the rest of the simulation examples, for the sake of smaller computational complexity, we adopt the suboptimal relay design in (45) based on Table II.

In our third example, we investigate the performance of the joint source and relay optimization algorithm in Table III at different iterations. We set $N_b = N = 2$, $M = 4$, and the $\mathrm{tr}(\mathbf{E}_1) + \mathrm{tr}(\mathbf{E}_2)$ objective is adopted. In particular, the source precoding matrices are updated by solving the subproblem (64)–(67). We observed in simulations that for most channel realizations, the algorithm converges within 10 iterations. The normalized SMSE of this algorithm after the first, second, and fifth iteration versus P_r is listed in Table IV. It can be seen that the difference between iterations is very small. Thus, in practice, only a small number of iterations are required to achieve a good performance.

In our fourth example, we study the performance of two-way MIMO relay systems based on the MSMI objective, the MSMSE objective, and the minimax MSE objective, respectively. We chose $N_b = N = 2$, $M = 6$, and for all objectives, we use the procedure listed in Table III. Fig. 5 shows the SMI of all three systems versus P_r. It can be seen from Fig. 5 that as expected, the MSMI-based relay design leads to a larger MI than the relay design using the MSMSE and minimax MSE criteria. The latter two systems have the same MI, since the unitary rotation matrices $\mathbf{U}_{b_1}$ and $\mathbf{U}_{b_2}$ do not change the system MI.

The uncoded BER of all three systems versus P_r is demonstrated in Fig. 6, where the QPSK constellations are used. The BER performance of the joint source and relay design algorithm in [6] based on the MSMSE criterion using the gradient

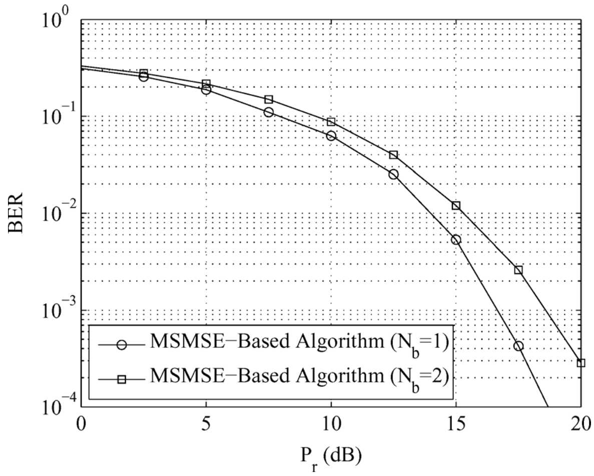

Fig. 7. Example 5: BER versus P_r with different N_b. $N = 2$, $M = 6$.

searching approach is also shown in Fig. 6. It can be seen from Fig. 6 that when the MSMSE objective is used, the proposed algorithm in Table III performs better than that in [6]. The reason is that the proposed algorithm exploits the optimal structure of the relay precoding matrix, while [6] does not. We also observe from Fig. 6 that the relay system designed under the minimax MSE criterion has the lowest BER, while the MSMI-based system has the highest BER. This is because MSMI is a good criterion only for coded systems in which the number of symbols for each coding block is very large. However, in the numerical comparison, we consider uncoded systems with a small number of symbols (QPSK, $N_b = 2$) for each block and compare the different schemes in term of raw BER. It is not surprising that the MSMI-based algorithm does not yield a better performance than algorithms based on the MSE in this setting. Minimax MSE is a better criterion for practical two-way MIMO relay systems with limited block length, as it yields a lower BER than other algorithms as shown in Fig. 6. This is due to the fact that BER is normally caused by the data stream having the highest MSE. In the minimax MSE-based system, all data streams have identical MSE, and thus, the system BER is reduced.

In the last example, we set $N = 2$, $M = 6$, and compare the BER performance of two-way MIMO relay systems with $N_b = 2$ and $N_b = 1$, respectively. The MSMSE objective is adopted. It can be seen from Fig. 7 that as expected, the system with $N_b = 1$ achieves a lower BER than the system with $N_b = 2$. However, we would like to mention that the system with $N_b = 2$ has twice data rate than the system with $N_b = 1$. This reflects the typical multiplexing-diversity tradeoff that exists in all MIMO communication systems. The proposed joint source and relay optimization algorithm is flexible in achieving such tradeoff since it works for any $1 \leq N_b \leq \min(M, N)$.

VI. Conclusion

We have derived the optimal structure of the source and relay precoding matrices for a two-way linear non-regenerative MIMO relay system with a broad class of frequently used objective functions. An iterative algorithm is developed to optimize the relay and source matrices. We have proposed a new suboptimal relay precoding matrix design which significantly reduces the computational complexity of the optimal design with only a marginal performance degradation. A novel minimax MSE-based relay system has been developed which has an improved BER performance compared with existing systems.

Appendix A
Proof of Theorem 1

Based on (22) and (23) we have

$$\mathbf{H}_{i,r} = \mathbf{U}_{2,i}\boldsymbol{\Sigma}_2\mathbf{V}_2^H, \quad i = 1,2, \quad \mathbf{H}_{r,1}\mathbf{B}_1 = \mathbf{U}_1\boldsymbol{\Sigma}_1\mathbf{V}_{1,2}^H$$
$$\mathbf{H}_{r,2}\mathbf{B}_2 = \mathbf{U}_1\boldsymbol{\Sigma}_1\mathbf{V}_{1,1}^H \quad (78)$$

where $\mathbf{U}_2 = [\mathbf{U}_{2,1}^T, \quad \mathbf{U}_{2,2}^T]^T$, $\mathbf{V}_1^H = [\mathbf{V}_{1,1}^H, \mathbf{V}_{1,2}^H]$, the dimensions of $\mathbf{U}_{2,1}$ and $\mathbf{U}_{2,2}$ are $N \times 2N$, and the dimensions of $\mathbf{V}_{1,1}$ and $\mathbf{V}_{1,2}$ are $N_b \times 2N_b$. Without loss of generality, $\mathbf{F}$ can be written as

$$\mathbf{F} = [\,\mathbf{V}_2 \quad \mathbf{V}_2^{\perp}\,]\begin{bmatrix}\mathbf{A} & \mathbf{K}\\ \mathbf{G} & \mathbf{J}\end{bmatrix}\begin{bmatrix}\mathbf{U}_1^H\\ (\mathbf{U}_1^{\perp})^H\end{bmatrix} \quad (79)$$

where $\mathbf{V}_2^{\perp}(\mathbf{V}_2^{\perp})^H = \mathbf{I}_M - \mathbf{V}_2\mathbf{V}_2^H$ and $\mathbf{U}_1^{\perp}(\mathbf{U}_1^{\perp})^H = \mathbf{I}_M - \mathbf{U}_1\mathbf{U}_1^H$ such that $[\mathbf{V}_2, \quad \mathbf{V}_2^{\perp}]$ and $[\mathbf{U}_1, \quad \mathbf{U}_1^{\perp}]$ are $M \times M$ unitary matrices. The dimensions of $\mathbf{A}$, $\mathbf{K}$, $\mathbf{G}$ and $\mathbf{J}$ are $2N \times 2N_b$, $2N \times (M-2N_b)$, $(M-2N) \times 2N_b$, and $(M-2N) \times (M-2N_b)$, respectively. Since $[\mathbf{V}_2, \quad \mathbf{V}_2^{\perp}]$ and $[\mathbf{U}_1, \quad \mathbf{U}_1^{\perp}]$ are $M \times M$ unitary matrices, for any $\mathbf{F}$, we have

$$\begin{bmatrix}\mathbf{A} & \mathbf{K}\\ \mathbf{G} & \mathbf{J}\end{bmatrix} = [\,\mathbf{V}_2, \quad \mathbf{V}_2^{\perp}\,]^H\,\mathbf{F}\,[\,\mathbf{U}_1, \quad \mathbf{U}_1^{\perp}\,].$$

Thus, using (79) to represent $\mathbf{F}$ does not lose any generality.

Substituting (78) and (79) back into (15), we obtain that $\mathbf{H}_{2,r}\mathbf{F}\mathbf{H}_{r,1}\mathbf{B}_1 = \mathbf{U}_{2,2}\boldsymbol{\Sigma}_2\mathbf{A}\boldsymbol{\Sigma}_1\mathbf{V}_{1,2}^H$ and $\mathbf{H}_{2,r}\mathbf{F}\mathbf{F}^H\mathbf{H}_{2,r}^H = \mathbf{U}_{2,2}\boldsymbol{\Sigma}_2(\mathbf{A}\mathbf{A}^H + \mathbf{K}\mathbf{K}^H)\boldsymbol{\Sigma}_2\mathbf{U}_{2,2}^H$. Thus, $\mathbf{E}_2^{\mathrm{o}}$ can be written as

$$\mathbf{E}_2^{\mathrm{o}} = \Big[\mathbf{I}_{N_b} + \mathbf{V}_{1,2}\boldsymbol{\Sigma}_1\mathbf{A}^H\boldsymbol{\Sigma}_2\mathbf{U}_{2,2}^H\left[\mathbf{U}_{2,2}\boldsymbol{\Sigma}_2(\mathbf{A}\mathbf{A}^H + \mathbf{K}\mathbf{K}^H)\right.$$
$$\left.\times\boldsymbol{\Sigma}_2\mathbf{U}_{2,2}^H + \mathbf{I}_N\right]^{-1}\mathbf{U}_{2,2}\boldsymbol{\Sigma}_2\mathbf{A}\boldsymbol{\Sigma}_1\mathbf{V}_{1,2}^H\Big]^{-1}. \quad (80)$$

Similarly, by substituting (78) and (79) back into (15) we obtain

$$\mathbf{E}_1^{\mathrm{o}} = \Big[\mathbf{I}_{N_b} + \mathbf{V}_{1,1}\boldsymbol{\Sigma}_1\mathbf{A}^H\boldsymbol{\Sigma}_2\mathbf{U}_{2,1}^H\left[\mathbf{U}_{2,1}\boldsymbol{\Sigma}_2(\mathbf{A}\mathbf{A}^H + \mathbf{K}\mathbf{K}^H)\right.$$
$$\left.\times\boldsymbol{\Sigma}_2\mathbf{U}_{2,1}^H + \mathbf{I}_N\right]^{-1}\mathbf{U}_{2,1}\boldsymbol{\Sigma}_2\mathbf{A}\boldsymbol{\Sigma}_1\mathbf{V}_{1,1}^H\Big]^{-1}. \quad (81)$$

Substituting (78) back into the left-hand-side of the transmission power constraint (21), we have

$$\mathrm{tr}\left(\mathbf{F}\left(\sum_{i=1}^{2}\mathbf{H}_{r,i}\mathbf{B}_i\mathbf{B}_i^H\mathbf{H}_{r,i}^H + \mathbf{I}_M\right)\mathbf{F}^H\right) = \mathrm{tr}\left(\mathbf{A}\left(\boldsymbol{\Sigma}_1^2\right.\right.$$
$$\left.\left.+\mathbf{I}_{2N_b}\right)\mathbf{A}^H + \mathbf{K}\mathbf{K}^H\right) + \mathrm{tr}\left(\mathbf{G}\left(\boldsymbol{\Sigma}_1^2 + \mathbf{I}_{2N_b}\right)\mathbf{G}^H + \mathbf{J}\mathbf{J}^H\right). \quad (82)$$

It can be clearly seen from (80) and (81) that $\mathbf{G}$ and $\mathbf{J}$ are irrelevant to $\mathbf{E}_1^{\circ}$ and $\mathbf{E}_2^{\circ}$. Since $q(\mathbf{d}[\mathbf{E}_i^{\circ}])$ is increasing in each one of the elements of $\mathbf{d}[\mathbf{E}_i^{\circ}]$, $q(\mathbf{d}[\mathbf{E}_1^{\circ}])$ and $q(\mathbf{d}[\mathbf{E}_2^{\circ}])$ are minimized if $\mathbf{K} = \mathbf{0}$. Moreover, from (82) we find that $\mathbf{K} = \mathbf{0}$, $\mathbf{G} = \mathbf{0}$, and $\mathbf{J} = \mathbf{0}$ minimize the transmit power consumption at the relay node. Thus, we have $\mathbf{F} = \mathbf{V}_2\mathbf{A}\mathbf{U}_1^H$.

Appendix B
Proof of Theorem 2

For the convenience of proof, we reproduce the equations for the objective function and the transmission power consumed by the relay node below

$$q\left(\mathbf{d}\left[\mathbf{E}_1^{\circ}(\mathbf{F})\right]\right) = q\left(\mathbf{d}\left[\left(\mathbf{I}_{N_b} + \mathbf{B}_2^H\mathbf{H}_{r,2}^H\mathbf{F}^H\mathbf{H}_{1,r}^H \times \left(\mathbf{I}_N + \mathbf{H}_{1,r}\mathbf{F}\mathbf{F}^H\mathbf{H}_{1,r}^H\right)^{-1}\mathbf{H}_{1,r}\mathbf{F}\mathbf{H}_{r,2}\mathbf{B}_2\right)^{-1}\right]\right) \quad (83)$$

$$q\left(\mathbf{d}\left[\mathbf{E}_2^{\circ}(\mathbf{F})\right]\right) = q\left(\mathbf{d}\left[\left(\mathbf{I}_{N_b} + \mathbf{B}_1^H\mathbf{H}_{r,1}^H\mathbf{F}^H\mathbf{H}_{2,r}^H \times \left(\mathbf{I}_N + \mathbf{H}_{2,r}\mathbf{F}\mathbf{F}^H\mathbf{H}_{2,r}^H\right)^{-1}\mathbf{H}_{2,r}\mathbf{F}\mathbf{H}_{r,1}\mathbf{B}_1\right)^{-1}\right]\right) \quad (84)$$

$$p_r = \operatorname{tr}\left(\mathbf{F}\left(\sum_{i=1}^{2}\mathbf{H}_{r,i}\mathbf{B}_i\mathbf{B}_i^H\mathbf{H}_{r,i}^H + \mathbf{I}_M\right)\mathbf{F}^H\right). \quad (85)$$

Let us introduce the following SVDs

$$\mathbf{H}_{r,2}\mathbf{B}_2 = \mathbf{U}_{1,1}\mathbf{\Sigma}_{1,1}\mathbf{V}_{1,1}^H, \quad \mathbf{H}_{r,1}\mathbf{B}_1 = \mathbf{U}_{1,2}\mathbf{\Sigma}_{1,2}\mathbf{V}_{1,2}^H \quad (86)$$

$$\mathbf{H}_{1,r} = \mathbf{U}_{2,1}\mathbf{\Sigma}_{2,1}\mathbf{V}_{2,1}^H, \quad \mathbf{H}_{2,r} = \mathbf{U}_{2,2}\mathbf{\Sigma}_{2,2}\mathbf{V}_{2,2}^H \quad (87)$$

where the dimensions of $\mathbf{U}_{1,i}, \mathbf{\Sigma}_{1,i}, \mathbf{V}_{1,i}, i = 1, 2$, are $M \times N_b$, $N_b \times N_b$, $N_b \times N_b$, respectively, and the dimensions of $\mathbf{U}_{2,i}$, $\mathbf{\Sigma}_{2,i}$, $\mathbf{V}_{2,i}$, $i = 1, 2$, are $N \times N$, $N \times N$, $M \times N$, respectively.

Case (a): Based on $\mathcal{R}(\mathbf{H}_{r,1}\mathbf{B}_1) \perp \mathcal{R}(\mathbf{H}_{r,2}\mathbf{B}_2)$ and $\mathcal{R}(\mathbf{H}_{1,r}^H) \perp \mathcal{R}(\mathbf{H}_{2,r}^H)$, we know from (86), (87) that $\mathbf{U}_{1,1} \perp \mathbf{U}_{1,2}$ and $\mathbf{V}_{2,1} \perp \mathbf{V}_{2,2}$. Thus, we have from (22), (23) that

$$\mathbf{U}_1 = [\,\mathbf{U}_{1,1}, \quad \mathbf{U}_{1,2}\,], \quad \mathbf{V}_2 = [\,\mathbf{V}_{2,1}, \quad \mathbf{V}_{2,2}\,] \quad (88)$$

$$\mathbf{\Sigma}_1 = \begin{bmatrix} \mathbf{\Sigma}_{1,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{1,2} \end{bmatrix}, \quad \mathbf{\Sigma}_2 = \begin{bmatrix} \mathbf{\Sigma}_{2,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{2,2} \end{bmatrix} \quad (89)$$

$$\mathbf{V}_1 = \begin{bmatrix} \mathbf{V}_{1,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{1,2} \end{bmatrix}, \quad \mathbf{U}_2 = \begin{bmatrix} \mathbf{U}_{2,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{2,2} \end{bmatrix}. \quad (90)$$

Now we can write $\mathbf{F}$ in (24) as

$$\mathbf{F} = [\,\mathbf{V}_{2,1}, \quad \mathbf{V}_{2,2}\,]\begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix}\begin{bmatrix} \mathbf{U}_{1,1}^H \\ \mathbf{U}_{1,2}^H \end{bmatrix} \quad (91)$$

where the dimension of $\mathbf{A}_{i,j}$ is $N \times N_b, i, j = 1, 2$. Substituting (91) back into (83)–(85), we obtain that

$$\mathbf{H}_{1,r}\mathbf{F}\mathbf{H}_{r,2}\mathbf{B}_2 = \mathbf{U}_{2,1}\mathbf{\Sigma}_{2,1}\mathbf{A}_{1,1}\mathbf{\Sigma}_{1,1}\mathbf{V}_{1,1}^H \quad (92)$$

$$\mathbf{H}_{1,r}\mathbf{F}\mathbf{F}^H\mathbf{H}_{1,r}^H = \mathbf{U}_{2,1}\mathbf{\Sigma}_{2,1}\left(\mathbf{A}_{1,1}\mathbf{A}_{1,1}^H + \mathbf{A}_{1,2}\mathbf{A}_{1,2}^H\right) \times \mathbf{\Sigma}_{2,1}\mathbf{U}_{2,1}^H \quad (93)$$

$$\mathbf{H}_{2,r}\mathbf{F}\mathbf{H}_{r,1}\mathbf{B}_1 = \mathbf{U}_{2,2}\mathbf{\Sigma}_{2,2}\mathbf{A}_{2,2}\mathbf{\Sigma}_{1,2}\mathbf{V}_{1,2}^H \quad (94)$$

$$\mathbf{H}_{2,r}\mathbf{F}\mathbf{F}^H\mathbf{H}_{2,r}^H = \mathbf{U}_{2,2}\mathbf{\Sigma}_{2,2}\left(\mathbf{A}_{2,1}\mathbf{A}_{2,1}^H + \mathbf{A}_{2,2}\mathbf{A}_{2,2}^H\right) \times \mathbf{\Sigma}_{2,2}\mathbf{U}_{2,2}^H \quad (95)$$

$$p_r = \operatorname{tr}\left(\mathbf{A}_{1,1}\left(\mathbf{\Sigma}_{1,1}^2 + \mathbf{I}_{N_b}\right)\mathbf{A}_{1,1}^H + \mathbf{A}_{1,2}\left(\mathbf{\Sigma}_{1,2}^2 + \mathbf{I}_{N_b}\right)\mathbf{A}_{1,2}^H + \mathbf{A}_{2,1}\left(\mathbf{\Sigma}_{1,1}^2 + \mathbf{I}_{N_b}\right)\mathbf{A}_{2,1}^H + \mathbf{A}_{2,2}\left(\mathbf{\Sigma}_{1,2}^2 + \mathbf{I}_{N_b}\right)\mathbf{A}_{2,2}^H\right). \quad (96)$$

It can be seen from (92)–(96) that (83)–(85) are minimized by $\mathbf{A}_{1,2} = \mathbf{A}_{2,1} = \mathbf{0}$. Thus, from (91) the optimal $\mathbf{F}$ is $\mathbf{F} = \mathbf{V}_2\begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{2,2} \end{bmatrix}\mathbf{U}_1^H$.

On the other hand, using (88)–(90), the relay precoding matrix in (45) can be rewritten as (97) shown at the bottom of the page. Obviously, (97) is optimal based on the analysis above.

Case (b): When $\mathcal{R}(\mathbf{H}_{r,1}\mathbf{B}_1) \parallel \mathcal{R}(\mathbf{H}_{r,2}\mathbf{B}_2), \mathcal{R}(\mathbf{H}_{1,r}^H) \perp \mathcal{R}(\mathbf{H}_{2,r}^H)$, we have $\mathbf{U}_{1,2} = \mathbf{U}_{1,1}$ in (86). It can be shown that in this case we have

$$\mathbf{U}_1 = [\,\mathbf{U}_{1,1}, \quad \mathbf{U}_{1,1}^{\perp}\,], \quad \mathbf{V}_2 = [\,\mathbf{V}_{2,1}, \quad \mathbf{V}_{2,2}\,] \quad (98)$$

$$\mathbf{\Sigma}_1 = \begin{bmatrix} \mathbf{\Xi} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{\Sigma}_2 = \begin{bmatrix} \mathbf{\Sigma}_{2,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{2,2} \end{bmatrix} \quad (99)$$

$$\mathbf{V}_1 = \begin{bmatrix} \mathbf{V}_{1,1}\mathbf{\Sigma}_{1,1}\mathbf{\Xi}^{-1} & \mathbf{V}_{1,1}\mathbf{\Sigma}_{1,2}\mathbf{\Xi}^{-1} \\ \mathbf{V}_{1,2}\mathbf{\Sigma}_{1,2}\mathbf{\Xi}^{-1} & -\mathbf{V}_{1,2}\mathbf{\Sigma}_{1,1}\mathbf{\Xi}^{-1} \end{bmatrix}$$

$$\mathbf{U}_2 = \begin{bmatrix} \mathbf{U}_{2,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{2,2} \end{bmatrix} \quad (100)$$

where $\mathbf{\Xi} \triangleq (\mathbf{\Sigma}_{1,1}^2 + \mathbf{\Sigma}_{1,2}^2)^{\frac{1}{2}}$. Now we can write terms in (83)–(85) as

$$\mathbf{H}_{1,r}\mathbf{F}\mathbf{H}_{r,2}\mathbf{B}_2 = \mathbf{U}_{2,1}\mathbf{\Sigma}_{2,1}\mathbf{A}_{1,1}\mathbf{\Sigma}_{1,1}\mathbf{V}_{1,1}^H \quad (101)$$

$$\mathbf{H}_{1,r}\mathbf{F}\mathbf{F}^H\mathbf{H}_{1,r}^H = \mathbf{U}_{2,1}\mathbf{\Sigma}_{2,1}\left(\mathbf{A}_{1,1}\mathbf{A}_{1,1}^H + \mathbf{A}_{1,2}\mathbf{A}_{1,2}^H\right) \times \mathbf{\Sigma}_{2,1}\mathbf{U}_{2,1}^H \quad (102)$$

$$\mathbf{H}_{2,r}\mathbf{F}\mathbf{H}_{r,1}\mathbf{B}_1 = \mathbf{U}_{2,2}\mathbf{\Sigma}_{2,2}\mathbf{A}_{2,1}\mathbf{\Sigma}_{1,2}\mathbf{V}_{1,2}^H \quad (103)$$

$$\mathbf{H}_{2,r}\mathbf{F}\mathbf{F}^H\mathbf{H}_{2,r}^H = \mathbf{U}_{2,2}\mathbf{\Sigma}_{2,2}\left(\mathbf{A}_{2,1}\mathbf{A}_{2,1}^H + \mathbf{A}_{2,2}\mathbf{A}_{2,2}^H\right) \times \mathbf{\Sigma}_{2,2}\mathbf{U}_{2,2}^H \quad (104)$$

$$p_r = \operatorname{tr}\left(\mathbf{A}_{1,1}(\mathbf{\Xi}^2 + \mathbf{I}_N)\mathbf{A}_{1,1}^H + \mathbf{A}_{1,2}\mathbf{A}_{1,2}^H + \mathbf{A}_{2,1}(\mathbf{\Xi}^2 + \mathbf{I}_N)\mathbf{A}_{2,1}^H + \mathbf{A}_{2,2}\mathbf{A}_{2,2}^H\right). \quad (105)$$

$$\mathbf{F} = \mathbf{V}_2\begin{bmatrix} \mathbf{\Sigma}_{2,1}^{-1}\mathbf{U}_{2,1}^H\mathbf{U}_{c_1}\mathbf{\Delta}_1\mathbf{\Pi}_1^{-1}\mathbf{R}_1^H\mathbf{V}_{1,1}\mathbf{\Sigma}_{1,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{2,2}^{-1}\mathbf{U}_{2,2}^H\mathbf{U}_{c_2}\mathbf{\Delta}_2\mathbf{\Pi}_2^{-1}\mathbf{R}_2^H\mathbf{V}_{1,2}\mathbf{\Sigma}_{1,2} \end{bmatrix}\mathbf{U}_1^H. \quad (97)$$

It can be seen from (101)–(105) that (83)–(85) are minimized by $\mathbf{A}_{1,2} = \mathbf{A}_{2,2} = \mathbf{0}$. Thus, from (91) the optimal $\mathbf{F}$ is $\mathbf{F} = \mathbf{V}_2 \begin{bmatrix} \mathbf{A}_{1,1} \\ \mathbf{A}_{2,1} \end{bmatrix} \mathbf{U}_{1,1}^H$. On the other hand, in this case, using (98)–(100) the relay precoding matrix in (45) can be rewritten as

$$\mathbf{F} = \mathbf{V}_2 \begin{bmatrix} \mathbf{\Sigma}_{2,1}^{-1}\mathbf{U}_{2,1}^H\mathbf{U}_{c_1}\mathbf{\Delta}_1\mathbf{\Pi}_1^{-1}\mathbf{R}_1^H\mathbf{V}_{1,1}\mathbf{\Sigma}_{1,1} \\ \mathbf{\Sigma}_{2,2}^{-1}\mathbf{U}_{2,2}^H\mathbf{U}_{c_2}\mathbf{\Delta}_2\mathbf{\Pi}_2^{-1}\mathbf{R}_2^H\mathbf{V}_{1,2}\mathbf{\Sigma}_{1,2} \end{bmatrix} \mathbf{U}_{1,1}^H$$

which is obviously optimal.

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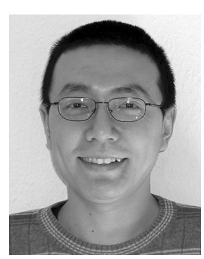

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